On the operad of bigraft algebras

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Abstract. In this paper, we study the notion of bigraft algebra, generalizing the notions of left and right graft algebras. We construct the free bigraft algebra on one generator in terms of certain planar rooted trees with decorated edges, and therefore describe explicitely the bigraft operad. We then compute its Koszul dual and show that the bigraft operad is Koszul. Moreover, we endow the free bigraft algebra on one generator with a universal Hopf algebra structure and a pairing. Finally, we prove an analogue of Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems. For this, we define the notion of infinitesimal bigraft bialgebras and we prove the existence of a new good triple of operads.

Résumé. Dans ce papier, nous étudions la notion d'algèbre bigreffe, généralisant les notions d'algèbres de greffes à gauche et à droite. Nous construisons l'algèbre bigreffe libre à un générateur en termes de certains arbres enracinés plans dont les arêtes sont décorées, et nous décrivons l'opérade bigreffe explicitement. Nous calculons alors son dual de Koszul et nous montrons que l'opérade bigreffe est Koszul. Par ailleurs, nous munissons l'algèbre bigreffe libre à un générateur d'une structure d'algèbre de Hopf universelle et d'un couplage. Enfin, nous prouvons un analogue des théorèmes de Poincaré-Birkhoff-Witt et de Cartier-Milnor-Moore. Pour cela, nous définissons la notion de bialgèbres bigreffes infinitésimales et nous prouvons l'existence d'un nouveau bon triplet d'opérades.

Keywords. Planar rooted trees, Operads, Koszul quadratic operads, Hopf algebras, Infinitesimal Hopf algebras.

AMS Classification. 05C05, 16W30, 18D50.

Contents

Introduction

In [Man12], the author constructs and studies a Hopf subalgebra of the Hopf algebra of ordered trees. He equips this subalgebra with two operations $\succ$ and $\prec$ (respectively the left graft and the right graft) and, after having defined the concept of bigraft algebra, he proves that this subalgebra is a bigraft algebra generated by one generator. Namely, a bigraft algebra is a $\mathbb{K}$-vector space A together with three $\mathbb{K}$-linear maps $*, \succ, \prec \colon A \otimes A \to A$ satisfying the following relations : for all $x, y, z \in A$,

$$\begin{aligned}
(x * y) \succ z &= x \succ (y \succ z), && (1)\\
(x \succ y) * z &= x \succ (y * z), && (2)\\
(x \prec y) \prec z &= x \prec (y * z), && (3)\\
(x * y) \prec z &= x * (y \prec z), && (4)\\
(x \succ y) \prec z &= x \succ (y \prec z), && (5)\\
(x * y) * z &= x * (y * z). && (6)
\end{aligned}$$

The aim of this paper is an algebraic study of the bigraft operad, denoted by $\mathcal{BG}$, associated to bigraft algebras. We will discribe this operad in terms of decorated rooted trees, show that it is Koszul and establish a new good triple of operads.

For this, we recall the notion of right graft algebras and left graft algebras (see [Foi10, Man12]). A right graft algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $*, \prec$: $A \otimes A \to A$ satisfying the relations (3), (4) and (6). With the relations (1), (2) and (6), we obtain the notion of left graft algebras. L. Foissy prove in [Foi10] that the operad of right graft algebras is given in terms of planar trees.

We prove in this paper that the bigraft operad is given in terms of subset of planar trees with their edges decorated with two possible decorations. This subset is defined in a recursive way by means of a universal B-operator. We denote by $\mathbf{H}$ the algebra generated by this subset.

Afterward, we compute the Koszul dual operad $\mathcal{BG}^!$ of the bigraft operad $\mathcal{BG}$. Considering a quotient of $\mathbf{H}$, we give a combinatorial description of the operad $\mathcal{BG}^!$ and a presentation of the homology of a bigraft algebra. We show that the operad of bigraft algebras is Koszul using Dotsenko-Khoroshkin's rewriting method (see [DK10, Hof10, LV12]).

In [Foi02, Hol03], the algebra of planar trees is equipped with a universal graft operator used to define a Hopf algebra structure. It is a noncommutative version of the Connes-Kreimer Hopf algebra of rooted trees (see [CK98, Moe01]). In the same way, we prove that the universal B-operator used before induces a Hopf algebra structure and a Hopf pairing on $\mathbf{H}$. We give the relationships between this coproduct and the bigraft products.

As these relationships do not permit to define a good notion of bigraft bialgebra, we consider another coproduct on $\mathbf{H}$, the deconcatenation coproduct $\tilde{\Delta}_{\mathcal{A}ss}$. We prove that the augmentation ideal of $\mathbf{H}$ is an infinitesimal bigraft bialgebra: this is a family $(A, *, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ where $*, \succ, \prec$: $A \otimes A \to A$, $\tilde{\Delta}_{\mathcal{A}ss} : A \to A \otimes A$, such that $(A, *, \succ, \prec)$ is a bigraft algebra and for all $x, y \in A$:

$$\begin{cases}
\tilde{\Delta}_{\mathcal{A}ss}(x * y) &= (x \otimes 1) * \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x) * (1 \otimes y) + x \otimes y,\\
\tilde{\Delta}_{\mathcal{A}ss}(x \succ y) &= (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y),\\
\tilde{\Delta}_{\mathcal{A}ss}(x \prec y) &= \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y).
\end{cases}$$

Then we show analogues of the Poincaré-Birkhoff-Witt theorem and of the Cartier-Milnor-Moore theorem for infinitesimal bigraft bialgebras using Loday's criterion (see [Lod08]). We prove that the primitive part $Prim(A)$ of an infinitesimal bigraft bialgebra A is a $\mathcal{L}$-algebra, that is to say a $\mathbb{K}$-vector space with two binary operations $\succ, \prec$ satisfying the entanglement relation (5) (see [Ler03, Ler08] for more details on $\mathcal{L}$-algebras). We construct a universal enveloping functor $U_{\mathcal{BG}}$ from $\mathcal{L}$-algebras to bigraft algebras. Then we prove the following result:

Theorem. *For any infinitesimal bigraft bialgebra A over a field $\mathbb{K}$, the following are equivalent:*
1. *A is a connected infinitesimal bigraft bialgebra,*
2. *A is cofree among the connected coalgebras,*
3. *A is isomorphic to $U_{\mathcal{BG}}(Prim(A))$ as an infinitesimal bigraft bialgebra.*

From this theorem and from Loday's criterion (see [Lod08]), we established a new good triple of operads $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ involving the operads of associative, bigraft and $\mathcal{L}$-algebras.

This paper is organised as follows: in the first section, we define bigraft algebras and we recall several facts on the algebra of planar trees and on right graft algebras. We give a combinatorial description of the free bigraft algebra and of the bigraft operad. Section 2 is devoted to the study of the Koszul dual of the bigraft operad. We compute the Koszul dual operad and show that the operad of bigraft algebras is Koszul. In section 3, we endow the free bigraft algebra on one generator with a universal Hopf algebra structure and a pairing. The last section deals with the notion of the infinitesimal bigraft bialgebras and we establish a new good triple of operads involving the operads of associative, bigraft and $\mathcal{L}$-algebras.

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Notations.

1. We shall denote by $\mathbb{K}$ a commutative field, of any characteristic. Every vector space, algebra, coalgebra, etc, will be taken over $\mathbb{K}$. Given a set X, we denote by $\mathbb{K}(X)$ the vector space spanned by X.

2. We shall denote by $T(V)$ the tensor algebra over a $\mathbb{K}$-vector space V. This is the tensor module
$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots$$
equipped with the concatenation.

3. Let V and W be two $\mathbb{K}$-vector spaces. We note $\tau : V \otimes W \to W \otimes V$ the unique $\mathbb{K}$-linear map called *the flip* such that $\tau(v \otimes w) = w \otimes v$ for all $v \otimes w \in V \otimes W$.

1 Operad of bigraft algebras

In this section, we define bigraft operad and we recall several facts on the algebra of planar trees and on right graft operad. Afterward, we prove that the bigraft operad is given in terms of subset of planar trees with their edges decorated with two possible decorations.

1.1 Bigraft algebras

The notions of left graft algebras and right graft algebras is introduced in [Foi10] by L. Foissy:

Definition 1 *1. A left graft algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $*, \succ: A\otimes A \to A$ respectively called product and left graft, satisfying the following relations: for all $x, y, z \in A$,*

$$\begin{array}{rcl}(x*y)*z &=& x*(y*z),\\ (x*y)\succ z &=& x\succ (y\succ z),\\ (x\succ y)*z &=& x\succ (y*z).\end{array}$$

2. The left graft operad, denoted by $\mathcal{LG}$, is the operad such that $\mathcal{LG}$-algebras are left graft algebras.

Definition 2 *1. A right graft algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $*, \prec: A\otimes A \to A$ respectively called product and right graft, satisfying the following relations: for all $x, y, z \in A$,*

$$\begin{array}{rcl}(x*y)*z &=& x*(y*z),\\ (x\prec y)\prec z &=& x\prec (y*z),\\ (x*y)\prec z &=& x*(y\prec z).\end{array}$$

2. The right graft operad, denoted by $\mathcal{RG}$, is the operad such that $\mathcal{RG}$-algebras are right graft algebras.

It is clear that the operads $\mathcal{LG}$ and $\mathcal{RG}$ are binary, quadratic, regular and set-theoretic (see [LV12] for a definition). It is proved that $\mathcal{LG}$ and $\mathcal{RG}$ are Koszul in [Foi09b]. We do not suppose that $\mathcal{LG}$-algebras and $\mathcal{RG}$-algebras have units for the product $*$. If A and B are two $\mathcal{LG}$-algebras, we say that a $\mathbb{K}$-linear map $f : A \to B$ is a $\mathcal{LG}$-morphism if $f(x*y) = f(x)*f(y)$ and $f(x\succ y) = f(x)\succ f(y)$ for all $x, y \in A$. We define in the same way the notion of $\mathcal{RG}$-morphism. We denote by $\mathcal{LG}$-alg the category of $\mathcal{LG}$-algebras and $\mathcal{RG}$-alg the category of $\mathcal{RG}$-algebras.

Remark. The category $\mathcal{LG}$-alg is equivalent to the category $\mathcal{RG}$-alg: let $(A, *, \succ)$ be a $\mathcal{LG}$-algebra, then $(A, *^\dagger, \succ^\dagger)$ is a $\mathcal{RG}$-algebra, where $x*^\dagger y = y*x$ and $x\succ^\dagger y = y\succ x$ for all $x, y \in A$. Note that $*^{\dagger\dagger} = *$ and $\succ^{\dagger\dagger}=\succ$. So we will only study the operad $\mathcal{RG}$.

We now give the definition of bigraft algebras:

Definition 3 *1. A bigraft algebra is a $\mathbb{K}$-vector space A together with three $\mathbb{K}$-linear maps $*, \succ, \prec: A\otimes A \to A$ satisfying the following relations: for all $x, y, z \in A$,*

$$\begin{array}{rcl}(x*y)\succ z &=& x\succ (y\succ z),\\ (x\succ y)*z &=& x\succ (y*z),\\ (x\prec y)\prec z &=& x\prec (y*z),\\ (x*y)\prec z &=& x*(y\prec z),\\ (x\succ y)\prec z &=& x\succ (y\prec z),\\ (x*y)*z &=& x*(y*z).\end{array} \tag{7}$$

2. The bigraft operad, denoted by $\mathcal{BG}$, is the operad such that $\mathcal{BG}$-algebras are bigraft algebras. $\mathcal{BG}$ is a binary, quadratic, regular and set-theoretic operad. We denote by $\widetilde{\mathcal{BG}}$ the nonsymmetric operad associated with the regular operad $\mathcal{BG}$.

In other words, a $\mathcal{BG}$-algebra $(A, *, \succ, \prec)$ is a $\mathcal{LG}$-algebra $(A, *, \succ)$ and a $\mathcal{RG}$-algebra $(A, *, \prec)$ verifying the so-called entanglement relation $(x\succ y)\prec z = x\succ (y\prec z)$ for all $x, y, z \in A$, that is to say $(A, \succ, \prec)$ is a $\mathcal{L}$-algebra (see definition 46). We do not suppose that $\mathcal{BG}$-algebras have a unit for the product $*$. If A and B are two $\mathcal{BG}$-algebras, a $\mathcal{BG}$-morphism from A to B is a $\mathbb{K}$-linear map $f : A \to B$ such that $f(x * y) = f(x) * f(y)$, $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by $\mathcal{BG}$-alg the category of $\mathcal{BG}$-algebras.

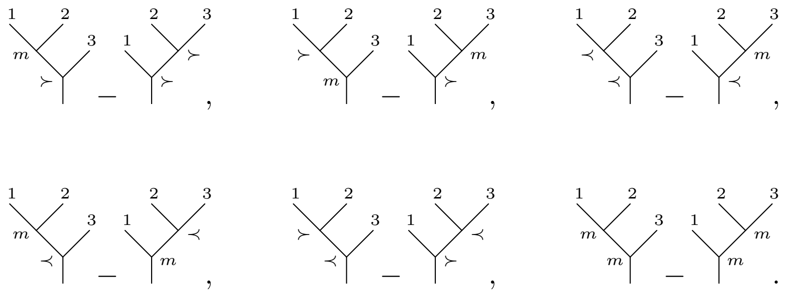
Remark. Let $(A, *, \succ, \prec)$ be a $\mathcal{BG}$-algebra. Then $(A, *^{\dagger}, \prec^{\dagger}, \succ^{\dagger})$ is a $\mathcal{BG}$-algebra, where $x *^{\dagger} y = y * x$, $x \succ^{\dagger} y = y \prec x$ and $x \prec^{\dagger} y = y \succ x$ for all $x, y \in A$.

From definition 3, we give a description of the operad $\mathcal{BG}$ by generators and relations:

Lemma 4 *The operad $\mathcal{BG}$ is the quadratic operad generated by three binary operations denoted by m, $\succ$ and $\prec$ and satisfying the following relations, where I is the unit element of $\mathcal{BG}$:*

$$\begin{cases} \succ \circ(m, I) - \succ \circ(I, \succ), \\ m \circ (\succ, I) - \succ \circ(I, m), \\ \prec \circ(\prec, I) - \prec \circ(I, m), \\ \prec \circ(m, I) - m \circ (I, \prec), \\ \prec \circ(\succ, I) - \succ \circ(I, \prec), \\ m \circ (m, I) - m \circ (I, m). \end{cases}$$

Remark. Graphically, the relations defining $\mathcal{BG}$ can be written in the following way:



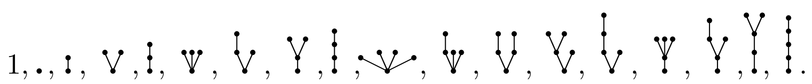
It is proved in [Foi10] that the right graft operad is given in terms of planar rooted trees. Using planar decorated trees, we will discribe the bigraft operad.

1.2 Free bigraft algebra and decorated planar rooted trees

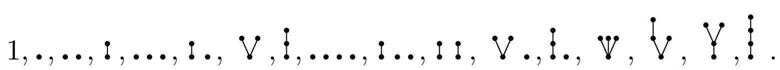
We briefly recall the construction of rooted trees (see [CK98]). A *rooted tree* is a finite graph, connected, without loops, with a distinguished vertex called the *root* [Sta02]. A *rooted forest* is a finite graph F such that any connected component of F is a rooted tree. The *length* of a forest F, denoted by $l(F)$, is the number of connected components of F. The set of vertices of the rooted forest F is denoted by $V(F)$. The *degree* of a forest F, denoted by $|F|$, is the number of its vertices.

Examples.

1. Rooted trees of degree ≤ 5:



2. Rooted forests of degree ≤ 4:



Let F be a rooted forest. The edges of F are oriented downwards (from the leaves to the roots). If $v, w \in V(F)$, we shall note $v \rightarrow w$ if there is an edge in F from v to w and $v \twoheadrightarrow w$ if there is an oriented path from v to w in F. By convention, $v \twoheadrightarrow v$ for any $v \in V(F)$. If T is a rooted tree and if $v \in V(T)$, we denote by $h(v)$ the *height* of v, that is to say the number of edges on the oriented path from v to the root of T. The *height* of a rooted forest F is $h(F) = \max(\{h(v),\ v \in V(F)\})$. We shall say that a tree T is a corolla if $h(T) \leq 1$.

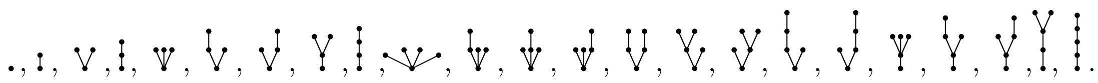
Let $\mathbf{H}_{CK}$ the $\mathbb{K}$-algebra generated by the set of rooted trees. Its product is given by the concatenation of rooted forests.

There exists a the noncommutative generalization of the Connes-Kreimer algebra [Foi02, Hol03].

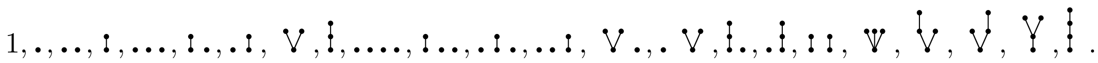
A *planar forest* is a rooted forest F such that the set of the roots of F is totally ordered and, for any vertex $v \in V(F)$, the set $\{w \in V(F) \mid w \rightarrow v\}$ is totally ordered. Planar forests are represented such that the total orders on the set of roots and the sets $\{w \in V(F) \mid w \rightarrow v\}$ for any $v \in V(F)$ is given from left to right. We denote by $\mathbb{T}_{NCK}$ the set of nonempty planar trees.

Examples.

1. Planar rooted trees of degree ≤ 5:



2. Planar rooted forests of degree ≤ 4:



We denote by $\mathbf{H}_{NCK}$ the algebra generated by planar trees. Its product is given by the concatenation of planar forests.

To describe the free bigraft algebra, we shall need decorated versions of the algebra $\mathbf{H}_{NCK}$. If $T \in \mathbf{H}_{NCK}$ is a tree, we shall note $E(T)$ the set of edges of T.

Definition 5 *Let $\mathcal{D}$ be a nonempty set.*

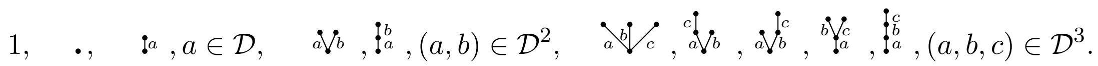
1. *A planar rooted tree with edges decorated by $\mathcal{D}$ is a couple (T, d), where T is a planar tree and $d : E(T) \longrightarrow \mathcal{D}$ is any map.*
2. *A planar rooted forest with edges decorated by $\mathcal{D}$ is a noncommutative monomial in planar rooted trees with edges decorated by $\mathcal{D}$.*

Let T be a planar rooted tree with edges decorated by $\mathcal{D}$. If $v, w \in V(T)$ and $d \in \mathcal{D}$, we shall denote $v \overset{d}{\rightarrow} w$ if there is an edge in T from v to w decorated by d.

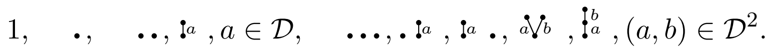
Note. This definition is different from [Foi02], where vertices of the trees are decorated.

Examples.

1. Planar rooted trees with edges decorated by $\mathcal{D}$ of degree smaller than 4:



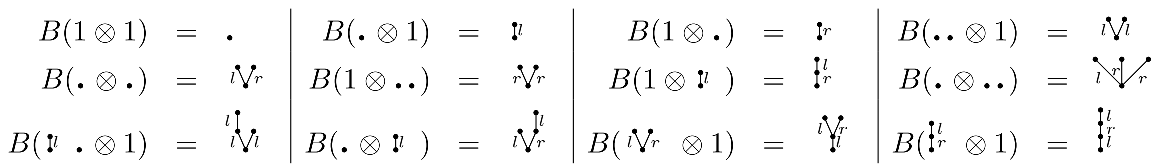
2. Planar rooted forests with edges decorated by $\mathcal{D}$ of degree smaller than 3:



Definition 6 *In the remainder of this article, we take $\mathcal{D} = \{l, r\}$.*

1. *We denote by $\mathbf{H}^{\mathcal{D}}_{NCK}$ the $\mathbb{K}$-algebra generated by planar trees with their edges decorated by $\mathcal{D}$. Its product is given by the concatenation.*
2. *We define an operator $B : \mathbf{H}^{\mathcal{D}}_{NCK} \otimes \mathbf{H}^{\mathcal{D}}_{NCK} \longrightarrow \mathbf{H}^{\mathcal{D}}_{NCK}$, which associates, to a tensor $F \otimes G$ of two forests $F, G \in \mathbf{H}^{\mathcal{D}}_{NCK}$, the tree obtained by grafting the roots of the trees of F and G on a common root and by decorating the edges between the common root and the roots of F by l and the edges between the common root and the roots of G by r.*

Examples.



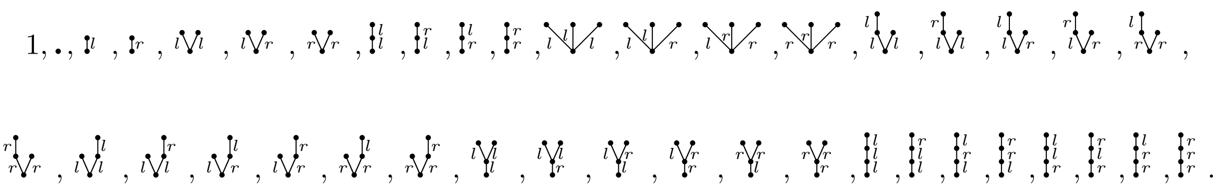
Definition 7 *We define a graded subset $\mathbb{T} = \cup_{n \geq 1} \mathbb{T}(n)$ of the set of trees of $\mathbf{H}^{\mathcal{D}}_{NCK}$ in the following inductive way :*

- *If $n = 1$, $\mathbb{T}(1) = \{\bullet\}$ is the set reduced to one element, the unique tree of degree 1.*
- *If $n \geq 2$, we suppose the set $\bigcup_{1 \leq k \leq n-1} \mathbb{T}(k)$ already constructed. Pick trees $F_1, \ldots, F_p$ and $G_1, \ldots, G_q \in \bigcup_{1 \leq k \leq n-1} \mathbb{T}(k)$ such that $|F_1| + \ldots + |F_p| + |G_1| + \ldots + |G_q| = n - 1$ and consider the new tree $B(F_1 \ldots F_p \otimes G_1 \ldots G_q)$.*

Let $\mathbb{F}(n)$ be the forests of degree n constructed with trees of $\mathbb{T}$ and $\mathbb{F} = \bigcup_{n \geq 0} \mathbb{F}(n)$ with $\mathbb{F}(0) = \{1\}$.

Remark. We also can define $\mathbb{T}$ as the smallest subset of $\mathbf{H}^{\mathcal{D}}_{NCK}$ such that $\bullet \in \mathbb{T}$ and which is closed under the action of the operator B.

Examples. Trees of $\mathbb{T}$ with degree ≤ 4:



Lemma 8 *Let $T \in \mathbf{H}^{\mathcal{D}}_{NCK}$ be a nonempty tree. Then $T \in \mathbb{T}$ if and only if for all $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$ and w_1 is left to w_2, $(d_1, d_2) \in \{(l, l), (l, r), (r, r)\}$, that is to say that we cannot have $(d_1, d_2) = (r, l)$.*

Proof. By induction on the degree n of T. If $n = 1$, $T = \bullet$ and it is trivial. Let $T \in \mathbf{H}^{\mathcal{D}}_{NCK}$ be a nonempty tree of degree $n \geq 2$ and assume that the property is true at rank $\leq n - 1$.

By construction, if $T \in \mathbb{T}$, there are $F_1, \ldots, F_p$ and $G_1, \ldots, G_q \in \mathbb{T}$ such that $T = B(F_1 \ldots F_p \otimes G_1 \ldots G_q)$ and $|F_i|, |G_i| \leq n-1$. Consider $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$

and w_1 is left to w_2. If v is not the root of T, $v \in V(F_i)$ or $v \in V(G_i)$ and by induction hypothesis $(d_1, d_2) \in \{(l,l),(l,r),(r,r)\}$. If v is the root of T, we have three cases:

1. w_1 is the root of F_i and w_2 is the root of F_j with $1 \leq i < j \leq p$. Then $(d_1, d_2) = (l,l)$.
2. w_1 is the root of F_i and w_2 is the root of G_j with $1 \leq i \leq p$ and $1 \leq j \leq q$. Then $(d_1, d_2) = (l,r)$.
3. w_1 is the root of G_i and w_2 is the root of G_j with $1 \leq i < j \leq q$. Then $(d_1, d_2) = (r,r)$.

In all cases, $(d_1, d_2) \in \{(l,l),(l,r),(r,r)\}$.

Reciprocally, let $T \in \mathbf{H}_{NCK}^{\mathcal{D}}$ be a nonempty tree such that for all $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$ and w_1 is left to w_2, $(d_1, d_2) \in \{(l,l),(l,r),(r,r)\}$. If v is the root of T, this condition implies that $T = B(F_1 \dots F_p \otimes G_1 \dots G_q)$ with $F_i, G_i \in \mathbf{H}_{NCK}^{\mathcal{D}}$ and $|F_i|, |G_i| \leq n-1$. Then F_i and G_i satisfy the condition of the decorations of the edges. Therefore $F_i, G_i \in \mathbb{T}$ by induction hypothesis and $T \in \mathbb{T}$. $\square$

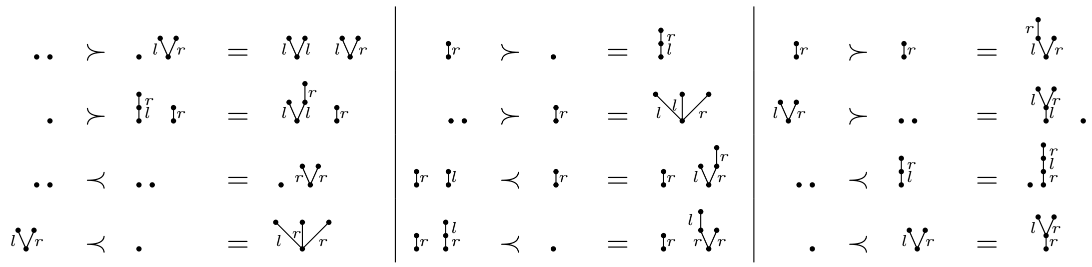
Definition 9 *We denote by* $\mathbf{H}$ *the subalgebra of* $\mathbf{H}_{NCK}^{\mathcal{D}}$ *generated by the trees of* $\mathbb{T}$ *and we denote by* $\mathbf{M}$ *the augmentation ideal of* $\mathbf{H}$*, that is* $\mathbf{M} = \overline{\mathbf{H}}$ *and* $\mathbf{H} = \mathbf{M}^+$.

Definition 10 *Let* $F, G \in \mathbb{F} \setminus \{1\}$*. Suppose that* $F = F_1 \dots F_n$*,* $G = G_1 \dots G_m$*, with* $F_i, G_i \in \mathbb{T}$*,* $F_1 = B(F_1^1 \otimes F_1^2)$ *and* $G_m = B(G_m^1 \otimes G_m^2)$*. We define :*

$$\begin{aligned} G \succ F &= B(GF_1^1 \otimes F_1^2)F_2 \dots F_n, \\ G \prec F &= G_1 \dots G_{m-1} B(G_m^1 \otimes G_m^2 F). \end{aligned}$$

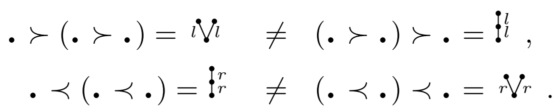
We extend $\succ, \prec : \mathbf{M} \otimes \mathbf{M} \to \mathbf{M}$ *by linearity.*

Examples.



In other terms, for $G \succ F$, G is grafted on the root of the first tree of F on the left with edges decorated by l. For $G \prec F$, F is grafted on the root of the last tree of G on the right with edges decorated by r. In particular, $F \succ \bullet = B(F \otimes 1)$ and $\bullet \prec F = B(1 \otimes F)$.

Remark. $\succ$ and $\prec$ are not associative:



Proposition 11 $\mathbf{M}$ *is given a graded* $\mathcal{BG}$*-algebra structure: for all* $x, y, z \in \mathbf{M}$*:*

$$(xy) \succ z = x \succ (y \succ z), \tag{8}$$
$$(x \succ y)z = x \succ (yz), \tag{9}$$
$$(x \prec y) \prec z = x \prec (yz), \tag{10}$$
$$(xy) \prec z = x(y \prec z), \tag{11}$$
$$(x \succ y) \prec z = x \succ (y \prec z). \tag{12}$$

Proof. We can restrict ourselves to $x, y, z \in \mathbb{F} \setminus \{1\}$. (9) and (11) are immediate. We put $x = x_1 B(x_2 \otimes x_3)$ and $z = B(z_1 \otimes z_2)z_3$ with $x_i, z_i \in \mathbb{F}$. Then

$$x \succ (y \succ z) = x \succ (B(yz_1 \otimes z_2)z_3) = B(xyz_1 \otimes z_2)z_3 = (xy) \succ (B(z_1 \otimes z_2)z_3) = (xy) \succ z,$$
$$(x \prec y) \prec z = (x_1 B(x_2 \otimes x_3 y)) \prec z = x_1 B(x_2 \otimes x_3 yz) = (x_1 B(x_2 \otimes x_3)) \prec (yz) = x \prec (yz).$$

So we have proved (8) and (10). In order to prove (12), we must study two cases :

1. If $y = B(y_1 \otimes y_2)$ is a tree,

$$(x \succ y) \prec z = B(xy_1 \otimes y_2) \prec z = B(xy_1 \otimes y_2 z) = x \succ B(y_1 \otimes y_2 z) = x \succ (y \prec z).$$

2. If $y = y_1 y_2$ with $y_1, y_2 \in \mathbb{F} \setminus \{1\}$,

$$(x \succ y) \prec z = ((x \succ y_1)y_2) \prec z = (x \succ y_1)(y_2 \prec z) = x \succ (y_1(y_2 \prec z)) = x \succ (y \prec z),$$

by (9) and (11).

$\square$

Remark. We can extend $\succ$ and $\prec$ to the vector space spanned by all decorated forests and we can proof that it is also a $\mathcal{BG}$-algebra. Note that a decorated forest can not always be constructed with the concatenation and the left and right grafts. For instance, the tree  can not be obtained with the operations $*, \succ$ and $\prec$. By cons, it is true for the forests of $\mathbf{M}$:

Theorem 12 *$(\mathbf{M}, m, \succ, \prec)$ is the free $\mathcal{BG}$-algebra generated by $\bullet$.*

Proof. Let A be a $\mathcal{BG}$-algebra and let $a \in A$. Let us prove that there exists a unique morphism of $\mathcal{BG}$-algebras $\phi : \mathbf{M} \to A$, such that $\phi(\bullet) = a$. We define $\phi(F)$ for any nonempty forest $F \in \mathbf{M}$ inductively on the degree of F by:

$$\left\{ \begin{array}{rcl} \phi(\bullet) & = & a, \\ \phi(F_1 \ldots F_k) & = & \phi(F_1) \ldots \phi(F_k) \text{ if } k \geq 2, \\ \phi(F) & = & \big(\phi(F^1) \succ a\big) \prec \phi(F^2) \text{ if } F = B(F^1 \otimes F^2) \text{ with } F^1, F^2 \in \mathbb{F}. \end{array} \right.$$

As the product of A is associative, this is perfectly defined. This map is linearly extended into a map $\phi : \mathbf{M} \to A$. Let us show that it is a morphism of $\mathcal{BG}$-algebras. By the second point, $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in \mathbf{M}$. Let F, G be two nonempty trees. Let us prove that $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$. Denote $F = B(F^1 \otimes F^2)$, $G = B(G^1 \otimes G^2)$ with F^1, F^2 and G^1, G^2 in $\mathbb{F}$. Then:

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,

$$\begin{array}{rcl} \phi(F \succ G) & = & \phi(B(FG^1 \otimes G^2)) \\ & = & \big(\phi(FG^1) \succ a\big) \prec \phi(G^2) \\ & = & \big(\phi(F) \succ \big(\phi(G^1) \succ a\big)\big) \prec \phi(G^2) \\ & = & \phi(F) \succ \big(\big(\phi(G^1) \succ a\big) \prec \phi(G^2)\big) \\ & = & \phi(F) \succ \phi(G). \end{array}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{array}{rcl} \phi(F \prec G) & = & \phi(B(F^1 \otimes F^2 G)) \\ & = & \big(\phi(F^1) \succ a\big) \prec \phi(F^2 G) \\ & = & \phi(F^1) \succ \big(a \prec \phi(F^2 G)\big) \\ & = & \phi(F^1) \succ \big(\big(a \prec \phi(F^2)\big) \prec \phi(G)\big) \\ & = & \big(\big(\phi(F^1) \succ a\big) \prec \phi(F^2)\big) \prec \phi(G) \\ & = & \phi(F) \prec \phi(G). \end{array}$$

If F, G are two nonempty forests, $F = F_1 \dots F_n$, $G = G_1 \dots G_m$. Then:

$$\begin{aligned} \phi(F \succ G) &= \phi((F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1)\dots))G_2 \dots G_m) \\ &= \phi(F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1)\dots))\phi(G_2 \dots G_m) \\ &= (\phi(F_1) \succ (\dots \phi(F_{n-1}) \succ (\phi(F_n) \succ \phi(G_1))\dots))\phi(G_2 \dots G_m) \\ &= (\phi(F_1 \dots F_n) \succ \phi(G_1))\phi(G_2 \dots G_m) \\ &= \phi(F) \succ \phi(G), \end{aligned}$$

and

$$\begin{aligned} \phi(F \prec G) &= \phi(F_1 \dots F_{n-1}(\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &= \phi(F_1 \dots F_{n-1})\phi((\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &= \phi(F_1 \dots F_{n-1})((\dots (\phi(F_n) \prec \phi(G_1)) \prec \phi(G_2) \dots) \prec \phi(G_m)) \\ &= \phi(F_1 \dots F_{n-1})(\phi(F_n) \prec \phi(G_1 \dots G_m)) \\ &= \phi(F) \prec \phi(G). \end{aligned}$$

So ϕ is a morphism of $\mathcal{BG}$-algebras.

Let $\phi' : \mathbf{M} \to A$ be another morphism of $\mathcal{BG}$-algebras such that $\phi'(\centerdot) = a$. Then for any planar trees $F_1, \dots, F_k$, $\phi'(F_1 \dots F_k) = \phi'(F_1) \dots \phi'(F_k)$. For any forests $F^1, F^2 \in \mathbf{M}$,

$$\begin{aligned} \phi'(B(F^1 \otimes F^2)) &= \phi'((F^1 \succ \centerdot) \prec F^2) \\ &= (\phi'(F^1) \succ \phi'(\centerdot)) \prec \phi'(F^2) \\ &= (\phi'(F^1) \succ a) \prec \phi'(F^2). \end{aligned}$$

So $\phi = \phi'$. □

Proposition 13 *Let $\dagger : \mathbf{H} \to \mathbf{H}$ be the $\mathbb{K}$-linear map built by induction as follows : $1^\dagger = 1$ and for all $F, G \in \mathbb{F}$, $(FG)^\dagger = G^\dagger F^\dagger$ and $(B(F \otimes G))^\dagger = B(G^\dagger \otimes F^\dagger)$. Then $\dagger$ is an involution over $\mathbf{H}$.*

Proof. Let us prove that $(F^\dagger)^\dagger = F$ for all $F \in \mathbb{F}$ by induction on the degree n of F. If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (B(H^\dagger \otimes G^\dagger))^\dagger = B((G^\dagger)^\dagger \otimes (H^\dagger)^\dagger) = B(G \otimes H) = F$, using the induction hypothesis for G and H.
2. If $F = GH$ is a forest, with $G, H \in \mathbb{F} \setminus \{1\}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (H^\dagger G^\dagger)^\dagger = (G^\dagger)^\dagger (H^\dagger)^\dagger = GH = F$, using again the induction hypothesis for G and H.

In all cases, $(F^\dagger)^\dagger = F$. □

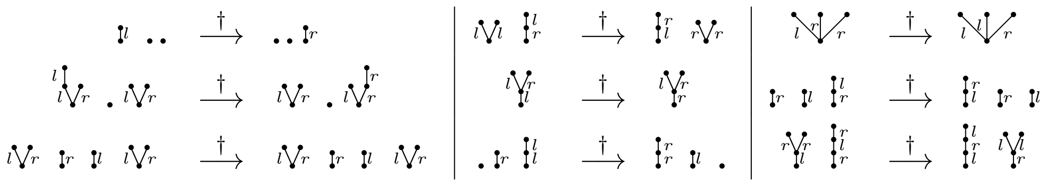
Remark.

1. Let $F \in \mathbf{H}$. The forest $F^\dagger$ is obtaiend by inverting the total orders on the set of roots of F and the sets $\{w \in V(F) \mid w \to v\}$ for any $v \in V(F)$ and by exchanging the decorations l and r.
2. We can rewrite the relations between $\dagger$ and the product m and B as follows:

$$\begin{aligned} m \circ (\dagger \otimes \dagger) &= \dagger \circ m \circ \tau, \\ B \circ (\dagger \otimes \dagger) &= \dagger \circ B \circ \tau. \end{aligned}$$

where τ is the flip defined in the introduction.

Examples.



The involution $\dagger : \mathbf{H} \to \mathbf{H}$ permits to exchange the structures of right and left graft algebra on $\mathbf{H}$:

Proposition 14 *For all forests $F, G \in \mathbf{M}$, $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ and $(G \prec F)^\dagger = F^\dagger \succ G^\dagger$.*

Proof. Let $F, G \in \mathbf{M}$ be two forests. We put $F = F_1 \ldots F_n$, $G = G_1 \ldots G_m$, $F_1 = B(F_1^1 \otimes F_1^2)$ and $G_m = B(G_m^1 \otimes G_m^2)$. Then

$$\begin{aligned}
(G \succ F)^\dagger &= (B(GF_1^1 \otimes F_1^2)F_2 \ldots F_n)^\dagger \\
&= F_n^\dagger \ldots F_2^\dagger B((F_1^2)^\dagger \otimes (GF_1^1)^\dagger) \\
&= F_n^\dagger \ldots F_2^\dagger B((F_1^2)^\dagger \otimes (F_1^1)^\dagger G^\dagger) \\
&= \left(F_n^\dagger \ldots F_2^\dagger B((F_1^2)^\dagger \otimes (F_1^1)^\dagger)\right) \prec G^\dagger \\
&= F^\dagger \prec G^\dagger.
\end{aligned}$$

Therefore $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ for all forests $F, G \in \mathbf{M}$. Moreover, as $\dagger$ is an involution over $\mathbf{M}$ (proposition 13), $(G \prec F)^\dagger = ((G^\dagger)^\dagger \prec (F^\dagger)^\dagger)^\dagger = ((F^\dagger \succ G^\dagger)^\dagger)^\dagger = F^\dagger \succ G^\dagger$. □

1.3 The bigraft operad

In this subsection, we give a combinatorial description of the bigraft operad and its dimension.

We denote by $f_n^{\mathbf{H}}$ the number of forests of degree n in $\mathbf{H}$, $t_n^{\mathbf{H}}$ the number of trees of degree n in $\mathbf{H}$ and $F_{\mathbf{H}}(x) = \sum_{n\geq 0} f_n^{\mathbf{H}} x^n$, $T_{\mathbf{H}}(x) = \sum_{n \geq 1} t_n^{\mathbf{H}} x^n$ the generating series associated. It is possible to calculate $F_{\mathbf{H}}(x)$ and $T_{\mathbf{H}}(x)$:

Proposition 15 *The generating series $F_{\mathbf{H}}$ and $T_{\mathbf{H}}$ are given by:*

$$\begin{aligned}
F_{\mathbf{H}}(x) &= \frac{3}{-1 + 4\cos^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right)}, \\
T_{\mathbf{H}}(x) &= \frac{4}{3}\sin^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right).
\end{aligned}$$

Proof. $\mathbf{H}$ is freely generated by the trees, therefore

$$F_{\mathbf{H}}(x) = \frac{1}{1 - T_{\mathbf{H}}(x)}. \tag{13}$$

We have the following relations:

$$\begin{aligned}
t_1^{\mathbf{H}} &= 1, \\
t_n^{\mathbf{H}} &= \sum_{k=1}^{n} \sum_{a_1 + \ldots + a_k = n-1} (k+1) t_{a_1}^{\mathbf{H}} \ldots t_{a_k}^{\mathbf{H}} \text{ if } n \geq 2.
\end{aligned}$$

Then

$$T_{\mathbf{H}}(x) - x = \sum_{k=1}^{\infty}(k+1)xT_{\mathbf{H}}(x)^k = xF \circ T_{\mathbf{H}}(x),$$

where $F(h) = \sum_{k=1}^{\infty}(k+1)h^k = \frac{2h-h^2}{(1-h)^2}$. We deduce the following equality:

$$T_{\mathbf{H}}(x)^3 - 2T_{\mathbf{H}}(x)^2 + T_{\mathbf{H}}(x) = x. \tag{14}$$

So $T_{\mathbf{H}}(x)$ is the inverse for the composition of $x^3 - 2x^2 + x$, that is to say

$$T_{\mathbf{H}}(x) = \frac{4}{3}\sin^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right).$$

Remark that, with (14), we obtain an inductive definition of the coefficients $t_n^{\mathbf{H}}$:

$$\begin{cases} t_1^{\mathbf{H}} = 1, \\ t_n^{\mathbf{H}} = 2\sum\limits_{i,j\geq 1 \text{ and } i+j=n} t_i^{\mathbf{H}}t_j^{\mathbf{H}} - \sum\limits_{i,j,k\geq 1 \text{ and } i+j+k=n} t_i^{\mathbf{H}}t_j^{\mathbf{H}}t_k^{\mathbf{H}} \text{ if } n \geq 2. \end{cases}$$

We deduce from (13) the equality

$$F_{\mathbf{H}}(x) = \frac{3}{-1 + 4\cos^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right)}.$$

□

This gives:

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{H}}$	1	2	7	30	143	728	3876	21318	120175	690690
$f_n^{\mathbf{H}}$	1	3	12	55	273	1428	7752	43263	246675	1430715

These are the sequences A006013 and A001764 in [Slo].

Recall that $\widetilde{\mathcal{BG}}$ is the nonsymmetric operad associated with the regular operad $\mathcal{BG}$. With the following proposition, we can identify $\mathcal{BG}$ with the vector space of nonempty forests $\in \mathbf{H}$.

Proposition 16 *For all $n \in \mathbb{N}^*$, $dim(\widetilde{\mathcal{BG}}(n)) = f_n^{\mathbf{H}}$ and the following map is bijective:*

$$\Psi : \begin{cases} \widetilde{\mathcal{BG}}(n) & \to & Vect(forests \in \mathbf{H} \text{ of degree } n) \subseteq \mathbf{M} \\ p & \to & p.(\bullet, \ldots, \bullet). \end{cases} \tag{15}$$

Proof. It suffices to show that Ψ is bijective. $(\mathbf{M}, m, \succ, \prec)$ is generated by $\bullet$ as $\mathcal{BG}$-algebra (with theorem 12) therefore Ψ is surjective. Ψ is injective by the freedom in theorem 12. □

In the remainder of this section, we identify $F \in \mathbb{F}(n)$ and $\Psi^{-1}(F) \in \mathcal{BG}(n)$.

Notations. In order to distinguish the composition in $\mathcal{BG}$ and the action of the operad $\mathcal{BG}$ on $\mathbf{M}$, we now denote by

1. $F \circ (F_1, \ldots, F_n)$ the composition of $\mathcal{BG}$.
2. $F \bullet (F_1, \ldots, F_n)$ the action of $\mathcal{BG}$ on $\mathbf{M}$.

In the following theorem, we describe the composition of $\mathcal{BG}$ in term of forests.

Theorem 17 *The composition of $\mathcal{BG}$ in the basis of forests belonging to $\mathbb{F}\setminus\{1\}$ can be inductively defined in this way:*

$$
\begin{array}{rcl}
\cdot \circ (H) & = & H,\\
FG\circ(H_1,\ldots,H_{|F|+|G|}) & = & F\circ(H_1,\ldots,H_{|F|})G\circ(H_{|F|+1},\ldots,H_{|F|+|G|}),\\
B(F\otimes G)\circ(H_1,\ldots,H_{|F|+|G|+1}) & = & \big((F\circ(H_1,\ldots,H_{|F|}))\succ H_{|F|+1}\big)\\
 & & \prec(G\circ(H_{|F|+2},\ldots,H_{|F|+|G|+1})).
\end{array}
$$

Proof. Note that $\Psi(\cdot)=\cdot=\Psi(I)$. Hence, $\cdot$ is the unit element of $\mathcal{BG}$.
By definition, $\Psi(\cdot\,\cdot)=\cdot\,\cdot=\Psi(m)$. So $\cdot\,\cdot=m$ in $\mathcal{BG}(2)$. Moreover, for all $F,G\in\mathbb{F}\setminus\{1\}$,

$$
\begin{array}{rcl}
\Psi(FG) & = & FG\\
 & = & m\bullet(F,G)\\
 & = & m\bullet(F\bullet(\cdot,\ldots,\cdot),G\bullet(\cdot,\ldots,\cdot))\\
 & = & (m\circ(F,G))\bullet(\cdot,\ldots,\cdot)\\
 & = & \Psi(m\circ(F,G)).
\end{array}
$$

So $FG=m\circ(F,G)=\cdot\,\cdot\circ(F,G)$.
We have $\Psi(\tdeux^{l})=\cdot\succ\cdot=\Psi(\succ)$. So $\tdeux^{l}=\succ$ in $\mathcal{BG}(2)$. Moreover, for all $F,G\in\mathbb{F}\setminus\{1\}$,

$$
\begin{array}{rcl}
\Psi(F\succ G) & = & F\succ G\\
 & = & \succ\bullet(F,G)\\
 & = & \succ\bullet(F\bullet(\cdot,\ldots,\cdot),G\bullet(\cdot,\ldots,\cdot))\\
 & = & (\succ\circ(F,G))\bullet(\cdot,\ldots.\cdot)\\
 & = & \Psi(\succ\circ(F,G)).
\end{array}
$$

So $F\succ G=\succ\circ(F,G)=\tdeux^{l}\circ(F,G)$.
As $\Psi(\tdeux^{r})=\cdot\prec\cdot=\Psi(\prec)$, $\tdeux^{r}=\prec$ in $\mathcal{BG}(2)$. Moreover, for all $F,G\in\mathbb{F}\setminus\{1\}$,

$$
\begin{array}{rcl}
\Psi(F\prec G) & = & F\prec G\\
 & = & \prec\bullet(F,G)\\
 & = & \prec\bullet(F\bullet(\cdot,\ldots,\cdot),G\bullet(\cdot,\ldots,\cdot))\\
 & = & (\prec\circ(F,G))\bullet(\cdot,\ldots.\cdot)\\
 & = & \Psi(\prec\circ(F,G)).
\end{array}
$$

So $F\prec G=\prec\circ(F,G)=\tdeux^{r}\circ(F,G)$.

Let $F\in\mathbb{F}(m)$ and $G\in\mathbb{F}(n)$ with $m,n\geq 1$. Let $H_1,\ldots,H_{m+n+1}\in\mathbb{F}\setminus\{1\}$. We will show that, in $\mathcal{BG}$,

$$
\begin{array}{rcl}
(FG)\circ(H_1,\ldots,H_{m+n}) & = & F\circ(H_1,\ldots,H_m)G\circ(H_{m+1},\ldots,H_{m+n})\\
B(F\otimes G)\circ(H_1,\ldots,H_{m+n+1}) & = & ((F\circ(H_1,\ldots,H_m))\succ H_{m+1})\\
 & & \prec(G\circ(H_{m+2},\ldots,H_{m+n+1})).
\end{array}
$$

Indeed, in $\mathcal{BG}$,

$$
\begin{array}{rcl}
(FG)\circ(H_1,\ldots,H_{m+n}) & = & (m\circ(F,G))\circ(H_1,\ldots,H_{m+n})\\
 & = & m\circ(F\circ(H_1,\ldots,H_m),G\circ(H_{m+1},\ldots,H_{m+n}))\\
 & = & F\circ(H_1,\ldots,H_m)G\circ(H_{m+1},\ldots,H_{m+n}),
\end{array}
$$

and

$$\begin{array}{rl} & B(F\otimes G)\circ(H_1,\ldots,H_{m+n+1})\\ = & ((F\succ \bullet)\prec G)\circ(H_1,\ldots,H_{m+n+1})\\ = & (\mathrm{I}^{r}\circ(\mathrm{I}^{l}\circ(F,\bullet),G))\circ(H_1,\ldots,H_{m+n+1})\\ = & \mathrm{I}^{r}\circ((\mathrm{I}^{l}\circ(F,\bullet))\circ(H_1,\ldots,H_{m+1}),G\circ(H_{m+2},\ldots,H_{m+n+1}))\\ = & \mathrm{I}^{r}\circ(\mathrm{I}^{l}\circ(F\circ(H_1,\ldots,H_m),\bullet\circ H_{m+1}),G\circ(H_{m+2},\ldots,H_{m+n+1}))\\ = & \mathrm{I}^{r}\circ((F\circ(H_1,\ldots,H_m))\succ H_{m+1},G\circ(H_{m+2},\ldots,H_{m+n+1}))\\ = & ((F\circ(H_1,\ldots,H_m))\succ H_{m+1})\prec(G\circ(H_{m+2},\ldots,H_{m+n+1})). \end{array}$$

□

Examples. Let $F_1,F_2,F_3\in\mathbb{F}\setminus\{1\}$.

$$\begin{array}{rcl|rcl} \bullet\bullet\circ(F_1,F_2) & = & F_1F_2 & {}^{l}\mathrm{V}^{r}\circ(F_1,F_2,F_3) & = & (F_1\succ F_2)\prec F_3\\ \mathrm{I}^{l}\circ(F_1,F_2) & = & F_1\succ F_2 & {}^{r}\mathrm{V}^{r}\circ(F_1,F_2,F_3) & = & F_1\prec(F_2F_3)\\ \mathrm{I}^{r}\circ(F_1,F_2) & = & F_1\prec F_2 & {}^{l}\mathrm{V}^{l}\circ(F_1,F_2,F_3) & = & (F_1F_2)\succ F_3\\ \bullet\,\mathrm{I}^{l}\circ(F_1,F_2,F_3) & = & F_1(F_2\succ F_3) & \mathrm{I}^{r}_{l}\circ(F_1,F_2,F_3) & = & (F_1\prec F_2)\succ F_3\\ \mathrm{I}^{r}\,\bullet\circ(F_1,F_2,F_3) & = & (F_1\prec F_2)F_3 & \mathrm{I}^{l}_{r}\circ(F_1,F_2,F_3) & = & F_1\prec(F_2\succ F_3)\\ \bullet\bullet\bullet\circ(F_1,F_2,F_3) & = & F_1F_2F_3 & \mathrm{I}^{l}_{l}\circ(F_1,F_2,F_3) & = & (F_1\succ F_2)\succ F_3 \end{array}$$

The free $\mathcal{BG}$-algebra over a vector space V is the $\mathcal{BG}$-algebra $\mathcal{BG}(V)$ such that any map from V to a $\mathcal{BG}$-algebra A has a natural extention as a $\mathcal{BG}$-morphism $\mathcal{BG}(V)\to A$. In other words the functor $\mathcal{BG}(-)$ is the left adjoint to the forgetful functor from $\mathcal{BG}$-algebras to vector spaces. Because the operad $\mathcal{BG}$ is regular, we get the following result :

Proposition 18 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{BG}$-algebra on V is*

$$\mathcal{BG}(V)=\bigoplus_{n\geq 1}\mathbb{K}\left(\mathbb{F}(n)\right)\otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F\in\mathbb{F}(n)$, $G\in\mathbb{F}(m)$, $v_1\otimes\ldots\otimes v_n\in V^{\otimes n}$ and $w_1\otimes\ldots\otimes w_m\in V^{\otimes m}$,

$$\begin{array}{rcl} (F\otimes v_1\otimes\ldots\otimes v_n)*(G\otimes w_1\otimes\ldots\otimes w_m) & = & (FG\otimes v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m),\\ (F\otimes v_1\otimes\ldots\otimes v_n)\succ(G\otimes w_1\otimes\ldots\otimes w_m) & = & (F\succ G\otimes v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m),\\ (F\otimes v_1\otimes\ldots\otimes v_n)\prec(G\otimes w_1\otimes\ldots\otimes w_m) & = & (F\prec G\otimes v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m). \end{array}$$

2 Koszul duality

2.1 Dual operad of the bigraft operad

As $\mathcal{BG}$ is a binary and quadratic operad, it admits a dual, in the sense of V. Ginzburg and M. Kapranov (see [GK94, MSS02]). We denote by $\mathcal{BG}^!$ the dual operad of $\mathcal{BG}$. We prove that $\mathcal{BG}^!$ is defined as follows:

Lemma 19 *The operad $\mathcal{BG}^!$, dual of $\mathcal{BG}$, is the quadratic operad generated by three operations m, $\succ$ and $\prec$ and satisfying the following relations, where I is the unit element of $\mathcal{BG}^!$:*

$$\left\{\begin{array}{l} r_1=\succ\circ(m,I)-\succ\circ(I,\succ),\\ r_2=m\circ(\succ,I)-\succ\circ(I,m),\\ r_3=\prec\circ(\prec,I)-\prec\circ(I,m),\\ r_4=\prec\circ(m,I)-m\circ(I,\prec),\\ r_5=\prec\circ(\succ,I)-\succ\circ(I,\prec),\\ r_6=m\circ(m,I)-m\circ(I,m), \end{array}\right.\quad \textit{and}\quad \left\{\begin{array}{l} r_7=\succ\circ(\succ,I),\\ r_8=\succ\circ(\prec,I),\\ r_9=m\circ(\prec,I),\\ r_{10}=\prec\circ(I,\prec),\\ r_{11}=\prec\circ(I,\succ),\\ r_{12}=m\circ(I,\succ). \end{array}\right. \tag{16}$$

The notations r_1 to r_{12} of the relations are introduced for future references.

Proof. Recall some notations. For any right Σ_n-module V, we denote by $V^!$ the right Σ_n-module $V^* \otimes (\varepsilon)$, where (ε) is the one-dimensional signature representation. Explicitly, the action of Σ_n on V^* is defined by $f^\sigma(x) = \varepsilon(\sigma) f(x^{\sigma^{-1}})$ for all $\sigma \in \Sigma_n$, $f \in V^!$, $x \in V$. The pairing between $V^!$ and V is given by: if $f \in V^!$, $x \in V$,

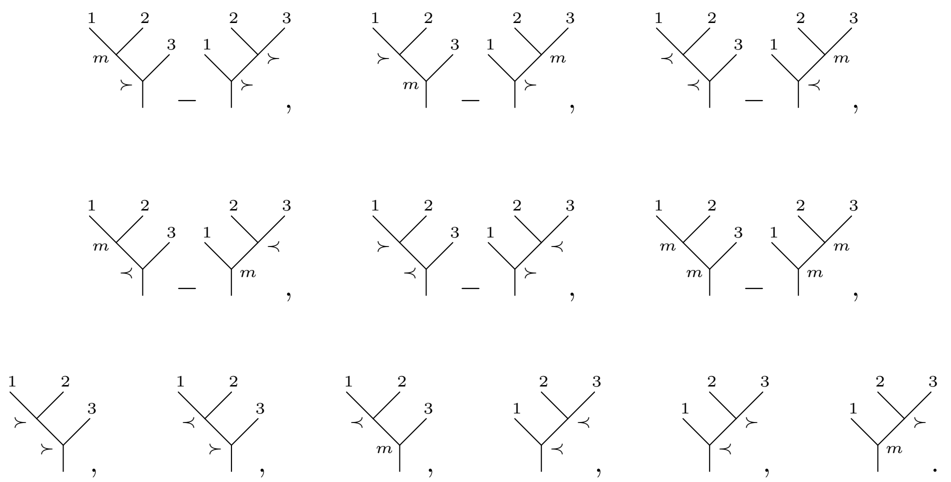
$$\langle -, - \rangle : V^! \otimes V \longrightarrow \mathbb{K}, \langle f, x \rangle = f(x), \tag{17}$$

and if $\sigma \in \Sigma_n$, $\langle f^\sigma, x^\sigma \rangle = \varepsilon(\sigma) \langle f, x \rangle$.

The operad $\mathcal{BG}$ is generated by the free Σ_2-module E, generated by m, $\succ$ and $\prec$, with relations $R \subseteq \mathcal{P}_E(3)$, where $\mathcal{P}_E(3)$ is the free operad generated by E and R is the free Σ_3-submodule of $\mathcal{P}_E(3)$ generated by $r_1, \ldots, r_6$. Note that $dim(E) = 6$, $dim(\mathcal{P}_E(3)) = 108$ and $dim(R) = 36$. So $\mathcal{BG}^!$ is generated by the dual $E^!$, with relations $R^\perp$ the annihilator of R for the pairing (17). So $dim(R^\perp) = dim(\mathcal{P}_E(3)) - dim(R) = 108 - 36 = 72$. We then verify that the given relations (16) for $\mathcal{BG}^!$ are indeed in $R^\perp$, that each of them generates a free Σ_3-module, and finally that there Σ_3-modules are in direct sum. So these relations entirely generate $R^\perp$. □

Remarks.

1. $\mathcal{BG}^!$ is a quotient of $\mathcal{BG}$.
2. $\mathcal{BG}^!$ is the symmetrization of the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$ generated by $m, \succ$ and $\prec$ and satisfying the relations r_1 to r_{12}.
3. Graphically, the relations defining $\mathcal{BG}^!$ can be written in the following way :



We now give a combinatorial description of the free $\mathcal{BG}^!$-algebra. We will show that it is a quotient of the free $\mathcal{BG}$-algebra $\mathbf{H}$.

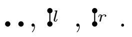
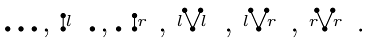
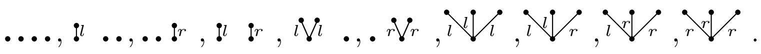
Definition 20 *Let $\mathbb{F}^!$ be the subset of forests of $\mathbb{F}$ containing the empty tree 1 and satisfying the following conditions : if $F_1 \ldots F_n \in \mathbb{F}^!$ with the F_i's nonempty trees, then*

1. *$F_1, \ldots, F_n$ are corollas $\in \mathbb{F}$,*
2. *if $\exists\ e \in E(F_i)$ decorated by l, then $i = 1$,*
3. *if $\exists\ e \in E(F_i)$ decorated by r, then $i = n$.*

We set $\mathbb{G} = \mathbb{F} \setminus \mathbb{F}^!$.

Remark. If $F_1 \dots F_n \in \mathbb{F}^!$ with $n \geq 2$, then $F_2, \dots, F_{n-1} = \bullet$, every $e \in E(F_1)$ is decorated by l and every $e \in E(F_n)$ is decorated by r. In particular, we have $\dagger(\mathbb{F}^!) = \mathbb{F}^!$, where $\dagger : \mathbf{H} \to \mathbf{H}$ is the involution defined in the proposition 13.

Examples. Forests of $\mathbb{F}^!$:

- In degree 1 : $\bullet$.
- In degree 2 : 
- In degree 3 : 
- In degree 4 : 

Let $\mathbf{H}^!$ be the $\mathbb{K}$-vector space spanned by $\mathbb{F}^!$ and $\mathbf{M}^!$ the $\mathbb{K}$-vector space spanned by $\mathbb{F}^! \setminus \{1\}$. We denote by $t_n^{\mathbf{H}^!}$ the number of trees of degree n in $\mathbf{H}^!$ and $f_n^{\mathbf{H}^!}$ the number of forests of degree n in $\mathbf{H}^!$. We put $T_{\mathbf{H}^!}(x) = \sum_{n \geq 1} t_n^{\mathbf{H}^!} x^n$ and $F_{\mathbf{H}^!}(x) = \sum_{n \geq 1} f_n^{\mathbf{H}^!} x^n$.

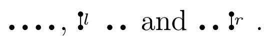
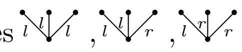
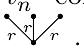
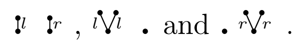
Proposition 21 *The generating series of $\mathbf{H}^!$ are given by :*

$$T_{\mathbf{H}^!}(x) = \frac{x}{(1-x)^2} \qquad \text{and} \qquad F_{\mathbf{H}^!}(x) = \frac{x}{(1-x)^3}.$$

Proof. We have $t_1^{\mathbf{H}^!} = 1, f_1^{\mathbf{H}^!} = 1$. For all $n \geq 2$, $t_n^{\mathbf{H}^!} = n$ because the corollas of degree n have $n-1$ edges with n possible different decorations. Moreover, for all $n \geq 2$,

$$f_n^{\mathbf{H}^!} = f_{n-1}^{\mathbf{H}^!} - t_{n-1}^{\mathbf{H}^!} + t_n^{\mathbf{H}^!} + (n-1)$$

where the term :

- $f_{n-1}^{\mathbf{H}^!} - t_{n-1}^{\mathbf{H}^!}$ corresponds to forests of degree n and of length ≥ 3 obtained from the forests of degree $n-1$ and of length ≥ 2 by adding $\bullet$ in the middle. For example, in degree 4, these are the forests 
- $t_n^{\mathbf{H}^!}$ corresponds to trees of degree n. In degree 4, these are the trees  and  .
- $n-1$ corresponds to forests of degree n and of length 2 obtained by product of a tree of degree k and a tree of degree $n-k$, $1 \leq k \leq n-1$. For example, in degree 4, these are the forests 

So $f_n^{\mathbf{H}^!} = f_{n-1}^{\mathbf{H}^!} + n = \frac{n(n+1)}{2}$. We deduce that $T_{\mathbf{H}^!}(x) = \frac{x}{(1-x)^2}$ and $F_{\mathbf{H}^!}(x) = \frac{x}{(1-x)^3}$. $\square$

Proposition 22 $(\mathbf{M}^!, m, \succ, \prec)$ *is a $\mathcal{BG}^!$-algebra.*

Proof. Let us prove that if $F, G \in \mathbb{F}$ are two nonempty forests such that $F \in \mathbb{G}$ or $G \in \mathbb{G}$, then $FG, F \succ G$ and $F \prec G \in \mathbb{G}$.

Suppose that $F \notin \mathbb{F}^!$. We have two cases :

1. If F is not a monomial of corollas. Then $h(F) \geq 2$. So $h(F \bullet G), h(G \bullet F) \geq 2$ and $F \bullet G \in \mathbb{G}, G \bullet F \in \mathbb{G}$ for all $\bullet \in \{m, \succ, \prec\}$.
2. If F is a monomial of corollas. As $F \notin \mathbb{F}^!$, $h(F) \geq 1$ and $F = F_1 F_2$ with F_1, F_2 nonempty such that $\exists\, e \in E(F_2)$ decorated by l (this is the same argument with r). Then $FG, GF \in \mathbb{G}$, $G \succ F = (G \succ F_1)F_2 \in \mathbb{G}$ and $F \prec G = F_1(F_2 \prec G) \in \mathbb{G}$. Moreover $h(F \succ G), h(G \prec F) \geq 2$ and $F \succ G, G \prec F \in \mathbb{G}$.

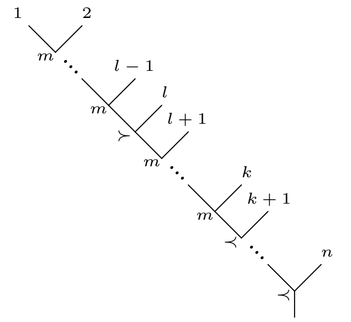
So $\mathbb{K}(\mathbb{G})$ is a $\mathcal{BG}$-ideal of the $\mathcal{BG}$-algebra $\mathbf{M}$ and the quotient vector space $\mathbf{M}^! = \mathbf{M}/\mathbb{K}(\mathbb{G})$ is a $\mathcal{BG}$-algebra. In the sequel, we shall identify a forest $F \in \mathbf{M}$ and its class in $\mathbf{M}^!$.

It remains to show relations $r7$ to $r12$. Let $F, G, H \in \mathbf{M}$ be three nonempty forests. $h((F \succ G) \succ H), h((F \prec G) \succ H), h(F \prec (G \prec H))$ and $h(F \prec (G \succ H)) \geq 3$. Therefore $(F \succ G) \succ H, (F \prec G) \succ H, F \prec (G \prec H), F \prec (G \succ H) \in \mathbb{G}$ and the relations $r7, r8, r10, r11$ are satisfied in $\mathbf{M}^!$. Moreover $(F \prec G)H, F(G \succ H) \in \mathbb{G}$ by considering the decorations of the edges and $r9, r12$ are true in $\mathbf{M}^!$. So $(\mathbf{M}^!, m, \succ, \prec)$ is a $\mathcal{BG}^!$-algebra. □

Theorem 23 *1. $(\mathbf{M}^!, m, \succ, \prec)$ is the free $\mathcal{BG}^!$-algebra generated by $\bullet$.*

2. For all $n \in \mathbb{N}^*$, $\mathcal{BG}^!(n) = \frac{n(n+1)!}{2}$.

3. For all $n \in \mathbb{N}^*$, $\mathcal{BG}^!(n)$ is freely generated, as a S_n-module, by the following trees:



(18)

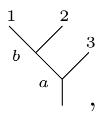
where $1 \leq l \leq k \leq n$.

Proof. Let $n \in \mathbb{N}^*$. Consider the following map :

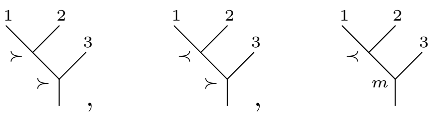
$$\Omega : \left\{ \begin{array}{rcl} \widetilde{\mathcal{BG}}^!(n) & \rightarrow & Vect(\text{forests } \in \mathbf{H}^! \text{ of degree } n) \subseteq \mathbf{M}^! \\ p & \rightarrow & p.(\bullet, \ldots, \bullet). \end{array} \right. \tag{19}$$

As $\mathbf{M}$ is generated as $\mathcal{BG}$-algebra by $\bullet$ (with theorem 12), $\mathbf{M}^!$ is also generated by $\bullet$ as $\mathcal{BG}$-algebra and therefore as $\mathcal{BG}^!$-algebra. So Ω is surjective and $dim(\mathcal{BG}^!(n)) \geq f_n^{\mathbf{H}^!} n! = \frac{n(n+1)!}{2}$.

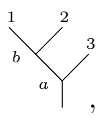
Moreover, from relations $r_1, \ldots, r_6$ and r_{10}, r_{11}, r_{12}, we obtain that $\mathcal{BG}^!(3)$ is generated by the trees of the following form:



with $a, b \in \{\prec, \succ, m\}$. Using relations r_7, r_8, r_9,

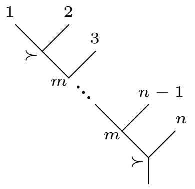


are eliminated. We deduce that $\mathcal{BG}^!(3)$ is generated by the trees of the following form:

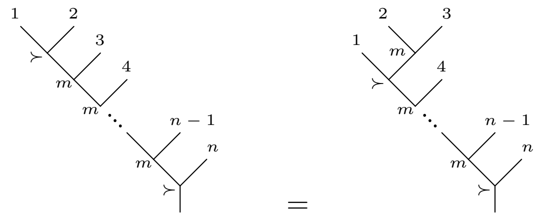


with $(a, b) \in \{(\succ, m), (m, \succ), (m, m), (\prec, \succ), (\prec, m), (\prec, \prec)\}$.

Let us prove that for all $n \geq 3$, the following tree is zero :



By induction on n. If $n = 3$ this tree is zero with the relation r_7. If $n \geq 4$,



with the relation r_2 and this tree is zero by induction hypothesis.

So $\mathcal{BG}^!(n)$ is generated, as a S_n-module, by the $\frac{n(n+1)}{2}$ trees given by the formula (18) and for all $n \in \mathbb{N}$, $dim(\mathcal{BG}^!(n)) \leq \dfrac{n(n+1)!}{2}$.

We deduce that $dim(\mathcal{BG}^!(n)) = \frac{n(n+1)!}{2}$. So Ω is bijective and $\mathbf{M}^!$ is the free $\mathcal{BG}^!$-algebra generated by $\boldsymbol{.\,.}$ Moreover $\mathcal{BG}^!(n)$ is freely generated, as a Σ_n-module, by the trees given to the formula (18), by equality of dimensions. $\square$

We give some numerical values:

n	1	2	3	4	5	6	7	8	9	10
$dim\left(\widetilde{\mathcal{BG}}^!(n)\right)$	1	3	6	10	15	21	28	36	45	55
$dim\left(\mathcal{BG}^!(n)\right)$	1	6	36	240	1800	15120	141120	1451520	16329600	199584000

These are the sequences A000217 and A001286 in [Slo].

Remark. We denote by $F_{\mathcal{BG}}(x)$ and $F_{\mathcal{BG}^!}(x)$ the generating series associated to operads $\mathcal{BG}$ and $\mathcal{BG}^!$. Using proposition 21, $F_{\mathcal{BG}^!}(x) = \frac{x}{(1-x)^3}$. Moreover, by proposition 15 (see the proof), $F_{\mathcal{BG}}(x) = \frac{T(x)}{1-T(x)}$ where $T(x) = (x - 2x^2 + x^3)^{-1}$. So $F_{\mathcal{BG}}^{-1}(x) = \frac{x}{(1+x)^3}$ and we have :

$$F_{\mathcal{BG}}\left(-F_{\mathcal{BG}^!}(-x)\right) = x.$$

This result is also a consequence of theorem 28.

As the map Ω defined in formula (19) is bijective, we can identify $F \in \mathbb{F}^!(n)$ and $\Omega^{-1}(F) \in \mathcal{BG}^!(n)$. We now describe the composition of $\mathcal{BG}^!$ in term of forests:

Theorem 24 *The composition $\circ$ of $\mathcal{BG}^!$ in the basis of forests belong to $\mathbb{F}^! \setminus \{1\}$ can be inductively defined in this way:*

$$\begin{array}{rcl} \bullet \circ (H) & = & H, \\ FG \circ (H_1, \ldots, H_{|F|+|G|}) & = & \left\{ \begin{array}{l} 0 \textit{ if the forest } H_1 \ldots H_{|F|+|G|} \notin \mathbb{F}^!, \\ F \circ (H_1, \ldots, H_{|F|})G \circ (H_{|F|+1}, \ldots, H_{|F|+|G|}) \textit{ if not,} \end{array} \right. \\ B(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \circ (H_1, \ldots, H_{p+q+1}) & = & \left\{ \begin{array}{l} 0 \textit{ if there exists } i \neq p+1 \textit{ such that } H_i \neq \bullet, \\ ((\underbrace{\bullet \ldots \bullet}_{p\times}) \succ H_{p+1}) \prec (\underbrace{\bullet \ldots \bullet}_{q\times}) \textit{ if not.} \end{array} \right. \end{array}$$

Proof. This is the same proof as for theorem 17. □

We denote by $\mathcal{BG}^!(V)$ the free $\mathcal{BG}^!$-algebra over a vector space V. Because the operad $\mathcal{BG}^!$ is regular, we get the following result:

Proposition 25 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{BG}^!$-algebra on V is*

$$\mathcal{BG}^!(V) = \bigoplus_{n\geq 1} \mathbb{K}\left(\mathbb{F}^!(n)\right) \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{F}^!(n)$, $G \in \mathbb{F}^!(m)$, $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \ldots \otimes w_m \in V^{\otimes m}$,

$$\begin{array}{rcl} (F \otimes v_1 \otimes \ldots \otimes v_n) * (G \otimes w_1 \otimes \ldots \otimes w_m) & = & (FG \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m), \\ (F \otimes v_1 \otimes \ldots \otimes v_n) \succ (G \otimes w_1 \otimes \ldots \otimes w_m) & = & (F \succ G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m), \\ (F \otimes v_1 \otimes \ldots \otimes v_n) \prec (G \otimes w_1 \otimes \ldots \otimes w_m) & = & (F \prec G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m). \end{array}$$

2.2 Homology of $\mathcal{BG}^!$-algebras

Definition 26 *By taking the elements of V homogenous of degree 1, $\mathcal{BG}^!(V)$ is naturally graduated. We define on $\mathcal{BG}^!(V)$ three coproducts in the following way: if $F = F_1 \ldots F_k \in \mathbb{F}^!(n)$ and $v = v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$, with*

- *if $k = 1$,*

$$F = B(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes \underbrace{\bullet \ldots \bullet}_{q\times}),$$

- *if $k \geq 2$,*

$$F_1 = B(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1), F_k = B(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \text{ and for all } 2 \leq i \leq k-1, F_i = \bullet,$$

with $p + q + k = n$, then

$$\begin{array}{rcl} \Delta(F \otimes v) & = & \displaystyle\sum_{i=0}^{k} (F_1 \ldots F_i \otimes v_1 \otimes \ldots \otimes v_{p+i}) \otimes (F_{i+1} \ldots F_k \otimes v_{p+i+1} \otimes \ldots \otimes v_n), \\ \Delta_{\succ}(F \otimes v) & = & \displaystyle\sum_{i=0}^{p} (\underbrace{\bullet \ldots \bullet}_{i\times} \otimes v_1 \otimes \ldots \otimes v_i) \otimes (B(\underbrace{\bullet \ldots \bullet}_{p-i\times} \otimes 1) F_2 \ldots F_k \otimes v_{i+1} \otimes \ldots \otimes v_n), \\ \Delta_{\prec}(F \otimes v) & = & \displaystyle\sum_{i=0}^{q} (F_1 \ldots F_{k-1} B(1 \otimes \underbrace{\bullet \ldots \bullet}_{q-i\times}) \otimes v_1 \otimes \ldots \otimes v_{n-i}) \otimes (\underbrace{\bullet \ldots \bullet}_{i\times} \otimes v_{n-i+1} \otimes \ldots \otimes v_n). \end{array}$$

Definition 27 *Let $(A, *, \succ, \prec)$ be a $\mathcal{BG}$-algebra. We define a differential $d : \mathcal{BG}^!(A)(n) \to \mathcal{BG}^!(A)(n-1)$ as the linear map uniquely determined by the following conditions :*

1. *For all $a \in A$, $d(\bullet \otimes a) = 0$.*
2. *For all $a, b \in A$, $d(\bullet\bullet \otimes a \otimes b) = a * b$, $d(\vdots^{l} \otimes a \otimes b) = a \succ b$ and $d(\vdots^{r} \otimes a \otimes b) = a \prec b$.*
3. *Let $\theta : \mathcal{BG}^!(A) \to \mathcal{BG}^!(A)$ be the following map:*

$$\theta : \left\{ \begin{array}{rcl} \mathcal{BG}^!(A) & \to & \mathcal{BG}^!(A) \\ x & \to & (-1)^{degree(x)} x \text{ for all homogeneous } x. \end{array} \right.$$

Then d is a θ-coderivation : for all $x \in \mathcal{BG}^!(A)$,

$$\begin{array}{rcl} \Delta(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta(x), \\ \Delta_\succ(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta_\succ(x), \\ \Delta_\prec(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta_\prec(x). \end{array}$$

So, d is the map which sends the element $(B(\underbrace{\bullet\ldots\bullet}_{p\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-2\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q\times})\otimes v_1\otimes\ldots\otimes v_n)$, where $p,q,k\in\mathbb{N}$, $k\geq 1$ and $p+q+k=n$ (if $k=1$, the element is $(B(\underbrace{\bullet\ldots\bullet}_{p\times}\otimes\underbrace{\bullet\ldots\bullet}_{q\times})\otimes v_1\otimes\ldots\otimes v_n)$), to

$$\begin{array}{rl} & \displaystyle\sum_{i=1}^{p-1}(-1)^{i-1}(B(\underbrace{\bullet\ldots\bullet}_{p-1\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-2\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q\times})\otimes v_1\otimes\ldots\otimes v_i * v_{i+1}\otimes\ldots\otimes v_n) \\ + & (-1)^{p-1}(B(\underbrace{\bullet\ldots\bullet}_{p-1\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-2\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q\times})\otimes v_1\otimes\ldots\otimes v_p\succ v_{p+1}\otimes\ldots\otimes v_n) \\ + & \displaystyle\sum_{i=p+1}^{p+k-1}(-1)^{i-1}(B(\underbrace{\bullet\ldots\bullet}_{p\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-3\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q\times})\otimes v_1\otimes\ldots\otimes v_i * v_{i+1}\otimes\ldots\otimes v_n) \\ + & (-1)^{p+k-1}(B(\underbrace{\bullet\ldots\bullet}_{p\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-2\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q-1\times})\otimes v_1\otimes\ldots\otimes v_{p+k}\prec v_{p+k+1}\otimes\ldots\otimes v_n) \\ + & \displaystyle\sum_{i=p+k+1}^{n-1}(-1)^{i-1}(B(\underbrace{\bullet\ldots\bullet}_{p\times}\otimes 1)\underbrace{\bullet\ldots\bullet}_{k-2\times}B(1\otimes\underbrace{\bullet\ldots\bullet}_{q-1\times})\otimes v_1\otimes\ldots\otimes v_i * v_{i+1}\otimes\ldots\otimes v_n). \end{array}$$

The homology of this complex will be denoted by $H_*(A)$. More clearly, for all $n\in\mathbb{N}$:

$$H_n(A) = \frac{Ker\left(d_{|\mathcal{BG}^!(A)(n+1)}\right)}{Im\left(d_{|\mathcal{BG}^!(A)(n+2)}\right)}$$

Examples. Let $a,b,c\in A$. Then $d(\bullet\otimes a)=0$, $d(\bullet\bullet\otimes a\otimes b)=a*b$, $d(|_l\otimes a\otimes b)=a\succ b$ and $d(|_r\otimes a\otimes b)=a\prec b$. In degree 3,

$$\begin{array}{rcl} d(\bullet\bullet\bullet\otimes a\otimes b\otimes c) & = & -(\bullet\bullet\otimes a\otimes(b*c))+(\bullet\bullet\otimes(a*b)\otimes c) \\ d(|_l\,\bullet\otimes a\otimes b\otimes c) & = & (\bullet\bullet\otimes(a\succ b)\otimes c)-(|_l\otimes a\otimes(b*c)) \\ d(\bullet\,|_r\otimes a\otimes b\otimes c) & = & (|_r\otimes(a*b)\otimes c)-(\bullet\bullet\otimes a\otimes(b\prec c)) \\ d({}^lV_l\otimes a\otimes b\otimes c) & = & -(|_l\otimes a\otimes(b\succ c))+(|_l\otimes(a*b)\otimes c) \\ d({}^lV_r\otimes a\otimes b\otimes c) & = & -(|_l\otimes a\otimes(b\prec c))+(|_r\otimes(a\succ b)\otimes c) \\ d({}^rV_r\otimes a\otimes b\otimes c) & = & -(|_r\otimes a\otimes(b*c))+(|_r\otimes(a\prec b)\otimes c) \end{array}$$

So, we obtain

$$\begin{array}{rcl} d^2(\bullet\bullet\bullet\otimes a\otimes b\otimes c) & = & -a*(b*c)+(a*b)*c \\ d^2(|_l\,\bullet\otimes a\otimes b\otimes c) & = & (a\succ b)*c-a\succ(b*c) \\ d^2(\bullet\,|_r\otimes a\otimes b\otimes c) & = & (a*b)\prec c-a*(b\prec c) \\ d^2({}^lV_l\otimes a\otimes b\otimes c) & = & -a\succ(b\succ c)+(a*b)\succ c \\ d^2({}^lV_r\otimes a\otimes b\otimes c) & = & -a\succ(b\prec c)+(a\succ c)\prec c \\ d^2({}^rV_r\otimes a\otimes b\otimes c) & = & -a\prec(b*c)+(a\prec b)\prec c \end{array}$$

Hence, the nullity of d^2 is equivalent to the six relations defining a $\mathcal{BG}$-algebra (see formula (7)). In particular :

$$H_0(A) = \frac{A}{A * A + A \succ A + A \prec A}$$

2.3 The bigraft operad is Koszul

In this section, we use the rewriting method described in [LV12] to prove that an operad is Koszul (see also [DK10, Hof10]). This is a short algorithmic method, based on the rewriting rules given by the relations.

Theorem 28 *The operad $\mathcal{BG}$ is Koszul.*

Proof. We consider $\widetilde{\mathcal{BG}}^! = \mathcal{P}(E, R)$ the nonsymmetric operad associated with $\mathcal{BG}^!$, where E is concentrated in degree 2 with $E(2) = \mathbb{K}m \oplus \mathbb{K} \succ \oplus \mathbb{K} \prec$ and R is concentrated in degree 3 with $R(3) \subseteq \mathcal{P}_E(3)$ the subspace generated by r_i, for all $i \in \{1, \ldots, 12\}$, defined in formula (16).

We use the lexicographical order and we set $\succ\, < m < \,\prec$. So, we can give the leading term for each r_i:

r_1	r_2	r_3	r_4	r_5	r_6
$\succ \circ (m, I)$	$m \circ (\succ, I)$	$\prec \circ (\prec, I)$	$\prec \circ (m, I)$	$\prec \circ (\succ, I)$	$m \circ (m, I)$

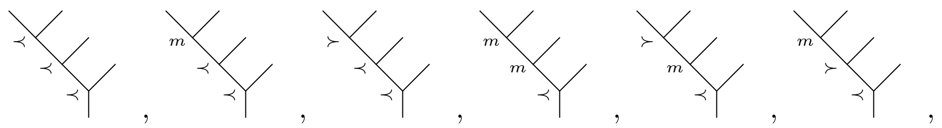
r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
$\succ \circ (\succ, I)$	$\succ \circ (\prec, I)$	$m \circ (\prec, I)$	$\prec \circ (I, \prec)$	$\prec \circ (I, \succ)$	$m \circ (I, \succ)$

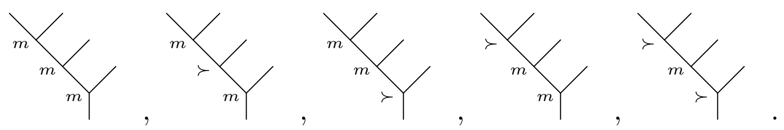
Observe that each relation gives rise to a rewriting rule in the operad $\widetilde{\mathcal{BG}}^!$:

$\succ \circ (m, I)$	$\mapsto$	$\succ \circ (I, \succ)$	$\succ \circ (\succ, I)$	$\mapsto$	0
$m \circ (\succ, I)$	$\mapsto$	$\succ \circ (I, m)$	$\succ \circ (\prec, I)$	$\mapsto$	0
$\prec \circ (\prec, I)$	$\mapsto$	$\prec \circ (I, m)$	$m \circ (\prec, I)$	$\mapsto$	0
$\prec \circ (m, I)$	$\mapsto$	$m \circ (I, \prec)$	$\prec \circ (I, \prec)$	$\mapsto$	0
$\prec \circ (\succ, I)$	$\mapsto$	$\succ \circ (I, \prec)$	$\prec \circ (I, \succ)$	$\mapsto$	0
$m \circ (m, I)$	$\mapsto$	$m \circ (I, m)$	$m \circ (I, \succ)$	$\mapsto$	0

Given three operations chosen in $\{\succ, m, \prec\}$, one can compose them along 5 different ways: they correspond to the 5 planar binary trees with 4 leaves. The monomial associated to a decorated planar binary tree is called critical if the two sub-trees with 3 leaves are leading terms.

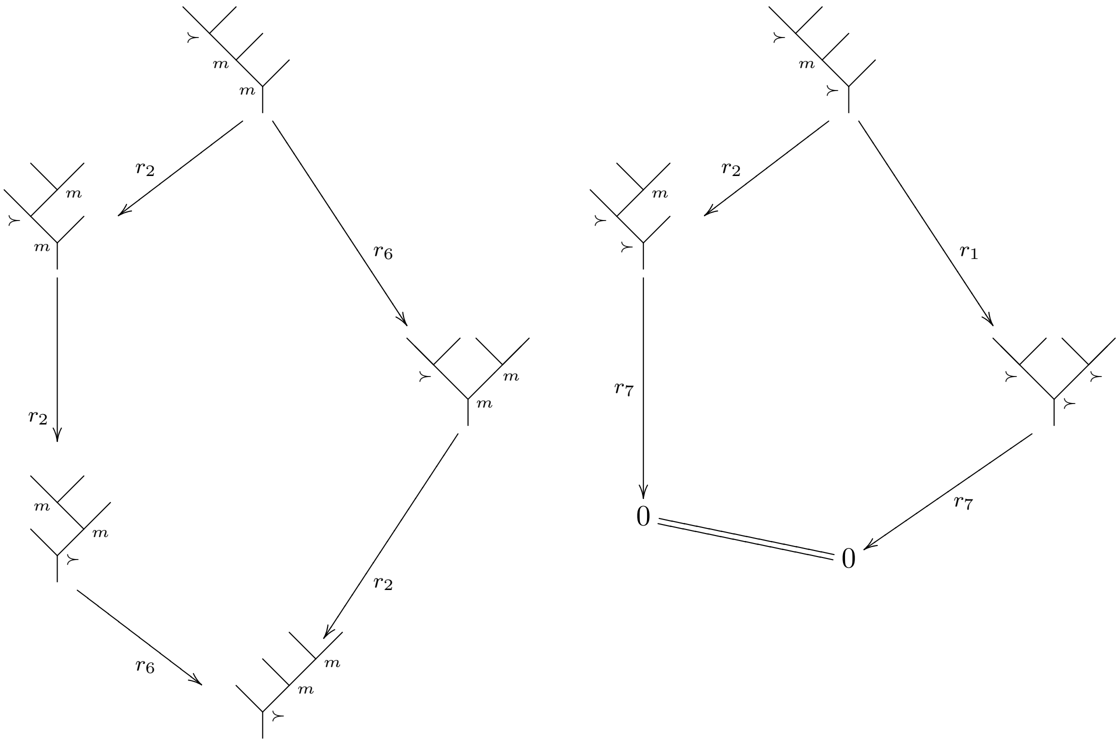
We have 11 criticals monomials:





There are at least two ways of rewriting a critical monomial, that is, until no more rewriting rules are applicable. If all these ways lead to the same element, then the critical monomial is said to be confluent.

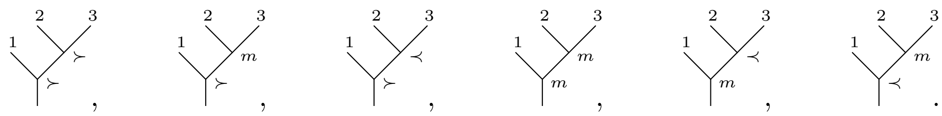
We can show that all critical monomials are confluent in the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$. We present for example the confluent graph for the last two criticals monomials:



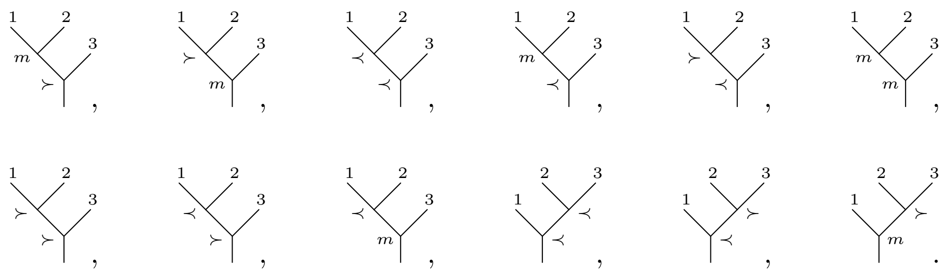
We can construct in the same way the confluent graph for each of the nine other criticals monomials.

Then, according to [LV12], we deduce that the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$ is Koszul. So the operad $\mathcal{BG}$ is Koszul. □

We can give the quadratic part of a PBW basis of $\mathcal{BG}^!$:



The quadratic part of a PBW basis of its dual $\left(\mathcal{BG}^!\right)^! = \mathcal{BG}$ is:



Recall that $H_*(.)$ is the homology of the complex defined in the section 2.2. Then we deduce of the previous theorem the following result:

Corollary 29 *Let $N \geq 1$ and $(A, *, \succ, \prec)$ be the free $\mathcal{BG}$-algebra generated by N elements. Then $H_0(A)$ is N-dimensional and if $n \geq 1$, $H_n(A) = (0)$.*

3 Hopf algebra structure on the free bigraft algebra

3.1 Hopf algebras of trees

Hopf algebra of rooted trees

A. Connes and D. Kreimer proved in [CK98] that the algebra $\mathbf{H}_{CK}$ generated by the set of rooted trees is a Hopf algebra. Its coproduct is defined by admissible cuts.

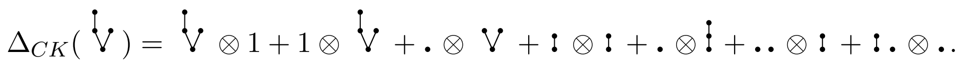
Let F be a rooted forest and $\boldsymbol{v}$ be a subset of $V(F)$. We shall say that $\boldsymbol{v}$ is an admissible cut of F, and we shall write $\boldsymbol{v} \models V(F)$, if $\boldsymbol{v}$ is totally disconnected, that is to say that $v \not\twoheadrightarrow w$ for any couple (v, w) of two different elements of $\boldsymbol{v}$. If $\boldsymbol{v} \models V(F)$, we denote by $Lea_{\boldsymbol{v}}(F)$ the rooted subforest of F obtained by keeping only the vertices above $\boldsymbol{v}$, that is to say $\{w \in V(F),\ \exists v \in \boldsymbol{v},\ w \twoheadrightarrow v\}$. Note that $\boldsymbol{v} \subseteq Lea_{\boldsymbol{v}}(F)$. We denote by $Roo_{\boldsymbol{v}}(F)$ the rooted subforest obtained by keeping the other vertices.

In particular, if $\boldsymbol{v} = \emptyset$, then $Lea_{\boldsymbol{v}}(F) = 1$ and $Roo_{\boldsymbol{v}}(F) = F$: this is the *empty cut* of F. If $\boldsymbol{v}$ contains all the roots of F, then it contains only the roots of F, $Lea_{\boldsymbol{v}}(F) = F$ and $Roo_{\boldsymbol{v}}(F) = 1$: this is the *total cut* of F. We shall write $\boldsymbol{v} \Vert\!\models V(F)$ if $\boldsymbol{v}$ is a nontotal, nonempty admissible cut of F.

Then $\mathbf{H}_{CK}$ is a Hopf algebra with the coproduct defined for any rooted forest F by:

$$\Delta_{CK}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \Vert\!\models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example:



Consider the operator $B_{CK} : \mathbf{H}_{CK} \longrightarrow \mathbf{H}_{CK}$, which associates, to a forest $F \in \mathbf{H}_{CK}$, the tree obtained by grafting the roots of the trees of F on a common root. For example, $B_{CK}(\ ) = B_{CK}(\ ) = $. It is shown in [CK98] that $(\mathbf{H}_{CK}, B_{CK})$ is an initial objet in the category of pairs (A, L), where A is a commutative algebra and $L : A \to A$ any linear operator. In other words, if A is a commutative algebra and if $L : A \to A$ is a linear map, then there exists a unique algebra morphism $\phi : \mathbf{H}_{CK} \to A$ such that $\phi \circ B_{CK} = L \circ \phi$.

Note that $\Delta_{CK} : \mathbf{H}_{CK} \longrightarrow \mathbf{H}_{CK} \otimes \mathbf{H}_{CK}$ is the unique algebra morphism such that

$$\Delta_{CK} \circ B_{CK} = B_{CK} \otimes 1 + (Id \otimes B_{CK}) \circ \Delta_{CK}.$$

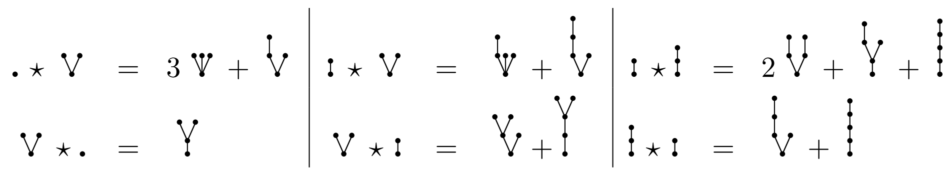
We here recall some results of [CL01].

Let $\mathbb{T}_{CK}$ be the set of the nonempty rooted trees of $\mathbf{H}_{CK}$ and $\mathbf{T}_{CK}$ the $\mathbb{K}$-vector space spanned by $\mathbb{T}_{CK}$.

If $T, T_1, T_2 \in \mathbb{T}_{CK}$, we denote by $n_{(T_1,T_2,T)}$ the number of admissible cuts $\boldsymbol{v}$ such that $Lea_{\boldsymbol{v}}(T) = T_1$ and $Roo_{\boldsymbol{v}}(T) = T_2$. We consider the linear map $\star : \mathbf{T}_{CK} \otimes \mathbf{T}_{CK} \to \mathbf{T}_{CK}$ such that, if $T_1, T_2 \in \mathbb{T}_{CK}$, then

$$T_1 \star T_2 = \sum_{T \in \mathbb{T}_{CK}} n_{(T_1,T_2,T)} T.$$

Examples.



Let us recall that a pre-Lie algebra is a $\mathbb{K}$-vector space A equipped with a binary operation $\star : A \otimes A \to A$ such that, for all $x, y, z \in A$,

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y).$$

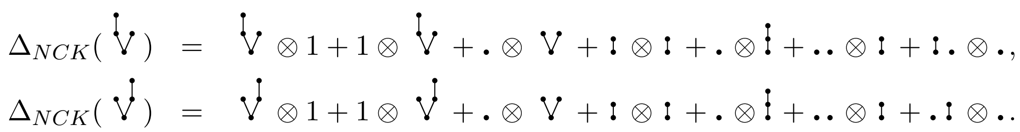
It is proved in [CL01] that $(\mathbf{T}_{CK}, \star)$ is the free pre-Lie algebra generated by $\bullet$.

Hopf algebra of planar rooted trees

The algebra of planar rooted trees $\mathbf{H}_{NCK}$ is also a Hopf algebra (see [Foi02]). If $\boldsymbol{v} \models V(F)$, then $Lea_{\boldsymbol{v}}(F)$ and $Roo_{\boldsymbol{v}}(F)$ are naturally planar forests. Then the coproduct of $\mathbf{H}_{NCK}$ is defined for any rooted forest F by:

$$\Delta_{NCK}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \Vdash V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example:

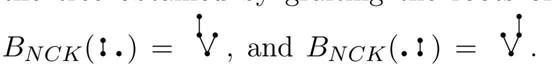


We have the same properties as in the commutative case: $(\mathbf{H}_{NCK}, B_{NCK})$ is an initial object in the category of pairs (A, L), where A is an algebra and $L : A \longrightarrow A$ any linear operator (see [Moe01]). Explicitely, let A be any algebra and let $L : A \longrightarrow A$ be a linear map. Then there exists a unique algebra morphism $\phi : \mathbf{H}_{NCK} \longrightarrow A$, such that $\phi \circ B_{NCK} = L \circ \phi$.
Moreover, $\Delta_{NCK} : \mathbf{H}_{NCK} \longrightarrow \mathbf{H}_{NCK} \otimes \mathbf{H}_{NCK}$ is the unique algebra morphism such that

$$\Delta_{NCK} \circ B_{NCK} = B_{NCK} \otimes 1 + (Id \otimes B_{NCK}) \circ \Delta_{NCK}.$$

We here recall some results of [Foi10].

Consider the operator $B_{NCK} : \mathbf{H}_{NCK} \longrightarrow \mathbf{H}_{NCK}$, which associates, to a forest $F \in \mathbf{H}_{NCK}$, the tree obtained by grafting the roots of the trees of F on a common root. For example,

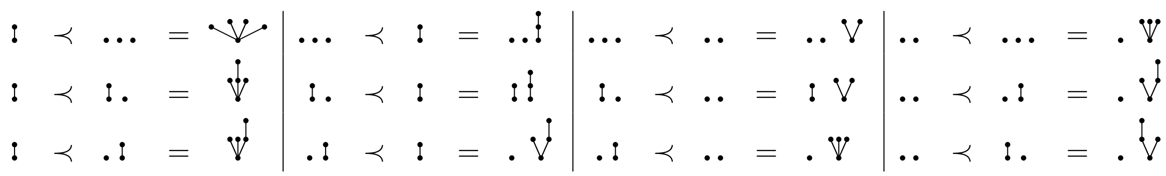


Let $\mathbf{M}_{NCK}$ be the $\mathbb{K}$-vector space spanned by the nonempty forests of $\mathbf{H}_{NCK}$. We consider the linear map $\prec : \mathbf{M}_{NCK} \otimes \mathbf{M}_{NCK} \to \mathbf{M}_{NCK}$ such that, if $F, G \in \mathbf{M}_{NCK}$ are two nonempty planar forests with $F = F_1 \dots F_n$ and $F_n = B_{NCK}(H)$, then

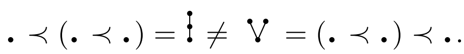
$$F \prec G = F_1 \dots F_{n-1} B_{NCK}(HG).$$

In other terms, G is grafted on the root of the last tree of F, on the right. In particular, we have $\bullet \prec G = B_{NCK}(G)$.

Examples.



Note that $\prec$ is not associative :



It is proved in [Foi10] that $(\mathbf{M}_{NCK}, m, \prec)$ is the free $\mathcal{RG}$-algebra generated by $\bullet$, where m is the concatenation product.

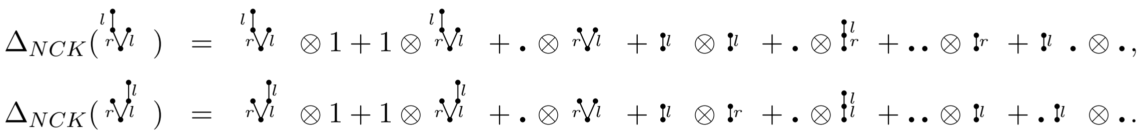
Hopf algebra of decorated planar rooted trees

In section 1.2, we have defined a decorated version $\mathbf{H}_{NCK}^{\mathcal{D}}$ of $\mathbf{H}_{NCK}$. We can define on $\mathbf{H}_{NCK}^{\mathcal{D}}$ a Hopf algebra structure generalizing the previous ones.

If $F \in \mathbf{H}_{NCK}^{\mathcal{D}}$ and $\boldsymbol{v} \models V(F)$, then $Lea_{\boldsymbol{v}}(F)$ and $Roo_{\boldsymbol{v}}(F)$ are naturally planar rooted forests with their edges decorated by $\mathcal{D}$. So $\mathbf{H}_{NCK}^{\mathcal{D}}$ is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any forest $F \in \mathbf{H}_{NCK}^{\mathcal{D}}$ by:

$$\Delta_{NCK}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \| \!\models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example:



Note that, unlike the cases of $(\mathbf{H}_{CK}, B_{CK})$ and $(\mathbf{H}_{NCK}, B_{NCK})$, this Hopf algebra structure on $\mathbf{H}_{NCK}^{\mathcal{D}}$ is not "universal" (with respect to the B-operator). We will show that:

- We can define a "universal" Hopf algebra structure on the free bigraft algebra on one generator $\mathbf{H}$ equipped with the operator B.
- With this Hopf algebra structure, $\mathbf{H}$ is a Hopf subalgebra of $\mathbf{H}_{NCK}^{\mathcal{D}}$.

3.2 Universal Hopf algebra on H

Let us prove that $(\mathbf{H}, B)$ is an initial object in the category of couples (A, L) where A is an unitary algebra and $L : A \otimes A \longrightarrow A$ any $\mathbb{K}$-linear map:

Lemma 30 *Let A be any unitary algebra and let $L : A \otimes A \to A$ be a $\mathbb{K}$-linear map. Then there exists a unique algebra morphism $\phi : \mathbf{H} \to A$ such that $\phi \circ B = L \circ (\phi \otimes \phi)$.*

Proof. *Existence.* We define an element $a_F \in A$ for any forest $F \in \mathbf{H}$ by induction on the degree of F as follows:

1. $a_1 = 1_A$.
2. If F is a tree, there exists two forests $G, H \in \mathbf{H}$ with $|G| + |H| = |F| - 1$ such that $F = B(G \otimes H)$; then $a_F = L(a_G \otimes a_H)$.
3. If F is not a tree, $F = F_1 \ldots F_n$ with $F_i \in \mathbb{T}$ and $|F_i| < |F|$; then $a_F = a_{F_1} \ldots a_{F_n}$.

Let $\phi : \mathbf{H} \to A$ be the unique linear morphism such that $\phi(F) = a_F$ for any forest F. Given two forests F and G, let us prove that $\phi(FG) = \phi(F)\phi(G)$. If $F = 1$ or $G = 1$, this is trivial. If not, $F = F_1 \ldots F_n$ and $G = G_1 \ldots G_m \neq 1$ and

$$\phi(FG) = a_{F_1} \ldots a_{F_n} a_{G_1} \ldots a_{G_m} = \phi(F)\phi(G).$$

Therefore ϕ is an algebra morphism. On the other hand, for all forests F, G,

$$\phi \circ B(F \otimes G) = a_{B(F \otimes G)} = L(a_F \otimes a_G) = L \circ (\phi \otimes \phi)(F \otimes G),$$

therefore $\phi \circ B = L \circ (\phi \otimes \phi)$.

Uniqueness. Let $\psi : \mathbf{H} \to A$ be an algebra morphism such that $\psi \circ B = L \circ (\psi \otimes \psi)$. Let us prove that $\psi(F) = a_F$ for any forest F by induction on the degree of F. If $F = 1$, $\psi(F) = 1_A = a_1$. If $|F| \geq 1$, we have two cases:

1. If F is a tree then $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$ and by induction hypothesis,
$$\psi(F) = \psi \circ B(G \otimes H) = L \circ (\psi \otimes \psi)(G \otimes H) = L(a_G \otimes a_H) = a_F.$$
2. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$ and by induction hypothesis,
$$\psi(F) = \psi(F_1) \dots \psi(F_n) = a_{F_1} \dots a_{F_n} = a_F.$$

So $\phi(F) = \psi(F)$ for any forest F and $\phi = \psi$. □

We now consider
$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{H} & \to & \mathbb{K} \\ F \in \mathbb{F} & \mapsto & \delta_{F,1}. \end{array} \right.$$
ε is an algebra morphism satisfying $\varepsilon \circ B = 0$.

Theorem 31 *Let $\Delta : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ be the unique algebra morphism such that $\Delta \circ B = L \circ (\Delta \otimes \Delta)$ with*
$$L : \left\{ \begin{array}{rcl} (\mathbf{H} \otimes \mathbf{H}) \otimes (\mathbf{H} \otimes \mathbf{H}) & \to & \mathbf{H} \otimes \mathbf{H} \\ (x \otimes y) \otimes (z \otimes t) & \mapsto & B(x \otimes z) \otimes \varepsilon(yt)1 + xz \otimes B(y \otimes t). \end{array} \right.$$
Then Δ is a coassociative coproduct and $(\mathbf{H}, m)$, endowed with this coproduct and the previous counit ε, is a graded connected Hopf algebra.

Proof. Thanks to lemma 30, Δ is well defined. To prove that $\mathbf{H}$ is a graded connected Hopf algebra, we must prove that Δ is coassociative, counitary, and homogeneous of degree 0.

We first show that ε is a counit for Δ. Let us prove that $(\varepsilon \otimes Id) \circ \Delta(F) = F$ for any forest $F \in \mathbb{F}$ by induction on the degree of F. If $F = 1$,
$$(\varepsilon \otimes Id) \circ \Delta(1) = \varepsilon(1)1 = 1.$$

We use the Sweedler's notation for $\mathbf{H}$: for all $x \in \mathbf{H}$, $\Delta(x) = \sum_x x^{(1)} \otimes x^{(2)}$. Then, if $G, H \in \mathbb{F}$,
$$\begin{aligned} \Delta(B(G \otimes H)) &= \sum_{G,H} B(G^{(1)} \otimes H^{(1)}) \otimes \varepsilon(G^{(2)} H^{(2)})1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \\ &= B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}). \end{aligned}$$
Note that, for all $x, y \in \mathbf{H}$,
$$\Delta \circ B(x \otimes y) = B(x \otimes y) \otimes 1 + (m \otimes B) \circ \Delta(x \otimes y),$$
with $\Delta(x \otimes y) = (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(x \otimes y)$ and τ the flip.

If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then,
$$\begin{aligned} (\varepsilon \otimes Id) \circ \Delta(F) &= (\varepsilon \otimes Id) \left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)} H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\ &= \varepsilon \circ B(G \otimes H) \otimes 1 + \sum_{G,H} \varepsilon(G^{(1)} H^{(1)}) B(G^{(2)} \otimes H^{(2)}) \\ &= 0 + \sum_{G,H} B(\varepsilon(G^{(1)}) G^{(2)} \otimes \varepsilon(H^{(1)}) H^{(2)}) \\ &= B(G \otimes H) \\ &= F. \end{aligned}$$

If $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$. As $(\varepsilon \otimes Id) \circ \Delta$ is an algebra morphism and using the induction hypothesis,

$$\begin{aligned}(\varepsilon \otimes Id) \circ \Delta(F) &= (\varepsilon \otimes Id) \circ \Delta(F_1) \dots (\varepsilon \otimes Id) \circ \Delta(F_n) \\ &= F_1 \dots F_n \\ &= F.\end{aligned}$$

Let us show that $(Id \otimes \varepsilon) \circ \Delta(F) = F$ for any forest $F \in \mathbb{F}$ by induction. If $F = 1$,

$$(Id \otimes \varepsilon) \circ \Delta(1) = 1\varepsilon(1) = 1.$$

If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$ and then

$$\begin{aligned}(Id \otimes \varepsilon) \circ \Delta(F) &= (Id \otimes \varepsilon)\left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)})\right) \\ &= B(G \otimes H) + \sum_{G,H} G^{(1)}H^{(1)}\varepsilon \circ B(G^{(2)} \otimes H^{(2)}) \\ &= B(G \otimes H) \\ &= F.\end{aligned}$$

If $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}$. As $(Id \otimes \varepsilon) \circ \Delta$ is an algebra morphism and using the induction hypothesis,

$$\begin{aligned}(Id \otimes \varepsilon) \circ \Delta(F) &= (Id \otimes \varepsilon) \circ \Delta(F_1) \dots (Id \otimes \varepsilon) \circ \Delta(F_n) \\ &= F_1 \dots F_n \\ &= F.\end{aligned}$$

Therefore ε is a counit for Δ.

Let us prove that Δ is coassociative. More precisely, we show that $(\Delta \otimes Id) \circ \Delta(F) = (Id \otimes \Delta) \circ \Delta(F)$ for any forest $F \in \mathbb{F}$ by induction on the degree of F. If $F = 1$ this is obvious. If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta(F) &= (\Delta \otimes Id) \circ \Delta \circ B(G \otimes H) \\ &= (\Delta \otimes Id)\left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)})\right) \\ &= B(G \otimes H) \otimes 1 \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \otimes 1 \\ &\quad + \sum_{G,H,G^{(1)},H^{(1)}} (G^{(1)})^{(1)}(H^{(1)})^{(1)} \otimes (G^{(1)})^{(2)}(H^{(1)})^{(2)} \otimes B(G^{(2)} \otimes H^{(2)}),\end{aligned}$$

$$\begin{aligned}(Id \otimes \Delta) \circ \Delta(F) &= (Id \otimes \Delta) \circ \Delta \circ B(G \otimes H) \\ &= (Id \otimes \Delta)\left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)})\right) \\ &= B(G \otimes H) \otimes 1 \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \otimes 1 \\ &\quad + \sum_{G,H,G^{(2)},H^{(2)}} G^{(1)}H^{(1)} \otimes (G^{(2)})^{(1)}(H^{(2)})^{(1)} \otimes B((G^{(2)})^{(2)} \otimes (H^{(2)})^{(2)}).\end{aligned}$$

We conclude with the coassociativity of Δ applied to G and H. If F is not a tree, $F = F_1 \ldots F_n$ with $F_i \in \mathbb{T}$. As $(\Delta \otimes Id) \circ \Delta$ and $(Id \otimes \Delta) \circ \Delta$ are algebra morphisms and using the induction hypothesis,

$$\begin{array}{rl} & (\Delta \otimes Id) \circ \Delta(F) \\ = & (\Delta \otimes Id) \circ \Delta(F_1) \ldots (\Delta \otimes Id) \circ \Delta(F_n) \\ = & \displaystyle\sum_{F_1,\ldots,F_n} \sum_{F_1^{(1)},\ldots,F_n^{(1)}} (F_1^{(1)})^{(1)} \ldots (F_n^{(1)})^{(1)} \otimes (F_1^{(1)})^{(2)} \ldots (F_n^{(1)})^{(2)} \otimes F_1^{(2)} \ldots F_n^{(2)} \\ = & \displaystyle\sum_{F_1,\ldots,F_n} \sum_{F_1^{(2)},\ldots,F_n^{(2)}} F_1^{(1)} \ldots F_n^{(1)} \otimes (F_1^{(2)})^{(1)} \ldots (F_n^{(2)})^{(1)} \otimes (F_1^{(2)})^{(2)} \ldots (F_n^{(2)})^{(2)} \\ = & (Id \otimes \Delta) \circ \Delta(F_1) \ldots (Id \otimes \Delta) \circ \Delta(F_n) \\ = & (Id \otimes \Delta) \circ \Delta(F). \end{array}$$

Therefore Δ is coassociative.

Let us show that Δ is homogeneous of degree 0. Easy induction, using the fact that L is homogeneous of degree 1. Note that it can also be proved using proposition 32. As it is graded and connected, it has an antipode denoted S. This ends the proof. □

We now give a combinatorial description of this coproduct:

Proposition 32 *Let $F \in \mathbb{F}$. Then*

$$\Delta(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

In other words, Δ is the restriction of $\Delta_{NCK} : \mathbf{H}_{NCK}^{\mathcal{D}} \rightarrow \mathbf{H}_{NCK}^{\mathcal{D}} \otimes \mathbf{H}_{NCK}^{\mathcal{D}}$ to $\mathbf{H}$ and $\mathbf{H}$ is a Hopf subalgebra of $\mathbf{H}_{NCK}^{\mathcal{D}}$.

Proof. Let Δ' be the unique algebra morphism from $\mathbf{H}$ to $\mathbf{H} \otimes \mathbf{H}$ defined by the formula of proposition 32. It is easy to show that, if $F, G \in \mathbb{F}$:

$$\left\{ \begin{array}{rcl} \Delta'(1) & = & 1 \otimes 1, \\ \Delta'(B(F \otimes G)) & = & B(F \otimes G) \otimes 1 + (m \otimes B) \circ \Delta'(F \otimes G). \end{array} \right.$$

By unicity in theorem 31, $\Delta' = \Delta$. □

Proposition 33 *For all forests $F \in \mathbf{H}$, $\Delta(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta(F)$.*

Proof. By induction on the degree n of F. If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}$ such that $|G|, |H| < n$. Then

$$\begin{array}{rcl} \Delta(F^\dagger) & = & \Delta(B(H^\dagger \otimes G^\dagger)) \\ & = & B(H^\dagger \otimes G^\dagger) \otimes 1 + (m \otimes B) \circ \Delta(H^\dagger \otimes G^\dagger) \\ & = & B(G \otimes H)^\dagger \otimes 1^\dagger \\ & & +(m \otimes B) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta) \otimes ((\dagger \otimes \dagger) \circ \Delta))(H \otimes G) \\ & = & (\dagger \otimes \dagger)(B(G \otimes H) \otimes 1) \\ & & +((m \circ (\dagger \otimes \dagger)) \otimes (B \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(H \otimes G) \\ & = & (\dagger \otimes \dagger)(B(G \otimes H) \otimes 1) \\ & & +(\dagger \otimes \dagger) \circ (m \otimes B) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(H \otimes G) \\ & = & (\dagger \otimes \dagger)\left(B(G \otimes H) \otimes 1 + (m \otimes B) \circ (Id \otimes \tau \otimes Id) \circ (\Delta \otimes \Delta)(G \otimes H)\right) \\ & = & (\dagger \otimes \dagger) \circ \Delta(F), \end{array}$$

using the induction hypothesis in the third equality.

2. If $F = GH$ is a forest, with $G, H \in \mathbb{F} \setminus \{1\}$ such that $|G|, |H| < n$. Then

$$\begin{aligned}
\Delta(F^\dagger) &= \Delta(H^\dagger G^\dagger) \\
&= (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H^\dagger) \otimes \Delta(G^\dagger)) \\
&= (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta(H)) \otimes ((\dagger \otimes \dagger) \circ \Delta(G))) \\
&= ((m \circ (\dagger \otimes \dagger)) \otimes (m \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H) \otimes \Delta(G)) \\
&= (\dagger \otimes \dagger) \circ (m \otimes m) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(H) \otimes \Delta(G)) \\
&= (\dagger \otimes \dagger) \circ (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta(G) \otimes \Delta(H)) \\
&= (\dagger \otimes \dagger) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis in the third equality.

In all cases, $\Delta(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta(F)$. □

3.3 A Hopf pairing on H

Using the B-operator, in this section we build a Hopf pairing on $\mathbf{H}$. Recall that a Hopf algebra pairing on a Hopf algebra (A, m_A, Δ_A, S_A) is a bilinear map $\langle -, - \rangle : A \times A \to \mathbb{K}$ such that: for all $x, x_1, x_2, y, y_1, y_2 \in A$,

$$\begin{aligned}
\langle x, m_A(y_1 \otimes y_2) \rangle &= \langle \Delta_A(x), y_1 \otimes y_2 \rangle, \\
\langle m_A(x_1 \otimes x_2), y \rangle &= \langle x_1 \otimes x_2, \Delta_A(y) \rangle, \\
\langle S_A(x), y \rangle &= \langle x, S_A(y) \rangle.
\end{aligned}$$

Definition 34 *Let $\gamma : \mathbb{K}(\mathbb{T}) \to \mathbb{K}(\mathbb{F}) \otimes \mathbb{K}(\mathbb{F})$ be the $\mathbb{K}$-linear map homogeneous of degree -1 such that for all $F, G \in \mathbb{F}$, $\gamma(B(F \otimes G)) = (-1)^{|G|} F \otimes G$ and Γ the $\mathbb{K}$-linear map defined by:*

$$\Gamma : \left\{ \begin{array}{rcl} \mathbf{H} & \to & \mathbf{H} \otimes \mathbf{H}, \\ T_1 \dots T_n & \to & \Delta(T_1) \dots \Delta(T_{n-1}) \gamma(T_n). \end{array} \right.$$

We note $\Delta(x) = \sum_x x^{(1)} \otimes x^{(2)}$ and $\Gamma(x) = \sum_x x_{(1)} \otimes x_{(2)}$. Then we define by induction on the degree a symmetric bilinear map $\langle -, - \rangle : \mathbf{H} \times \mathbf{H} \to \mathbb{K}$ with the following assertions:

1. *for all $x \in \mathbf{H}$, $\langle 1, x \rangle = \varepsilon(x)$,*
2. *for all $x, y, z \in \mathbf{H}$, $\langle xy, z \rangle = \sum_z \left\langle x, z^{(1)} \right\rangle \left\langle y, z^{(2)} \right\rangle$,*
3. *for all $x, y, z \in \mathbf{H}$, $\langle B(x \otimes y), z \rangle = \sum_z \langle x, z_{(1)} \rangle \langle y, z_{(2)} \rangle$.*

Theorem 35 *The bilinear map $\langle -, - \rangle : \mathbf{H} \times \mathbf{H} \to \mathbb{K}$ is a symmetric Hopf algebra pairing. In addition, if x and y are homogeneous of different degrees, then $\langle x, y \rangle = 0$.*

To prove this theorem, we first give two useful lemmas.

Lemma 36 *Let A be a Hopf algebra and $L : A \otimes A \to A$ a linear morphism such that for all $a, b \in A$,*

$$\Delta \circ L(a \otimes b) = L(a \otimes b) \otimes 1 + (m \otimes L) \circ \Delta(a \otimes b).$$

Then the unique algebra morphism $\phi : \mathbf{H} \to A$ such that $\phi \circ B = L \circ (\phi \otimes \phi)$ (lemma 30) is a Hopf algebra morphism.

Proof. Let us prove that $\Delta \circ \phi(F) = (\phi \otimes \phi) \circ \Delta(F)$ for any forest $F \in \mathbf{H}$ by induction on the degree of F. If $F = 1$, this is obvious. If F is a tree, $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$. Then

$$\begin{aligned}
\Delta \circ \phi(F) &= \Delta \circ \phi \circ B(G \otimes H) \\
&= \Delta \circ L \circ (\phi \otimes \phi)(G \otimes H) \\
&= L(\phi(G) \otimes \phi(H)) \otimes 1 + \sum_{\phi(G),\phi(H)} (\phi(G))^{(1)}(\phi(H))^{(1)} \otimes L((\phi(G))^{(2)} \otimes (\phi(H))^{(2)}) \\
&= \phi \circ B(G \otimes H) \otimes \phi(1) + \sum_{G,H} \phi(G^{(1)}H^{(1)}) \otimes \phi \circ B(G^{(2)} \otimes H^{(2)}) \\
&= (\phi \otimes \phi) \left(B(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B(G^{(2)} \otimes H^{(2)}) \right) \\
&= (\phi \otimes \phi) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis for the fourth equality. If F is not a tree, $F = F_1 \dots F_n$ and then

$$\begin{aligned}
\Delta \circ \phi(F) &= \Delta \circ \phi(F_1) \dots \Delta \circ \phi(F_n) \\
&= (\phi \otimes \phi) \circ \Delta(F_1) \dots (\phi \otimes \phi) \circ \Delta(F_n) \\
&= (\phi \otimes \phi) \circ \Delta(F),
\end{aligned}$$

using the induction hypothesis for the second equality and $\Delta \circ \phi$ and $(\phi \otimes \phi) \circ \Delta$ are algebra morphisms for the first and third equality.

Let us prove that $\varepsilon \circ \phi = \varepsilon$. As $\varepsilon \circ \phi$ and ε are algebra morphisms, it suffices to show that $\varepsilon \circ \phi(F) = \varepsilon(F)$ for any tree $F \in \mathbb{T}$. If $F = 1$, this is obvious. If $F = B(G \otimes H)$ with $G, H \in \mathbb{F}$, $\varepsilon(F) = 0$ and

$$\begin{aligned}
\varepsilon \circ \phi(F) &= \varepsilon \circ \phi \circ B(G \otimes H) \\
&= \varepsilon \circ L \circ (\phi \otimes \phi)(G \otimes H).
\end{aligned}$$

Let us prove that $\varepsilon \circ L = 0$: for all $a, b \in A$,

$$\begin{aligned}
\Delta \circ L(a \otimes b) &= L(a \otimes b) \otimes 1 + \sum_{a,b} a^{(1)}b^{(1)} \otimes L(a^{(2)} \otimes b^{(2)}) \\
(\varepsilon \otimes Id) \circ \Delta \circ L(a \otimes b) &= \varepsilon \circ L(a \otimes b)1 + \sum_{a,b} \varepsilon(a^{(1)}b^{(1)})L(a^{(2)} \otimes b^{(2)}) \\
&= \varepsilon \circ L(a \otimes b) + L(a \otimes b) \\
&= L(a \otimes b)
\end{aligned}$$

Therefore $\varepsilon \circ L = 0$ and $\varepsilon \circ \phi = \varepsilon$. □

Notation. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a graded vector space. We denote by $V^{\circledast} = \bigoplus_{n=0}^{\infty} V_n^*$ the graded dual of V. If H is a graded Hopf algebra, $H^{\circledast}$ is also a graded Hopf algebra.

Lemma 37 *Let $\Gamma^* : \mathbf{H}^{\circledast} \otimes \mathbf{H}^{\circledast} \to \mathbf{H}^{\circledast}$ be the transpose of Γ (Γ is introduced in definition 34). Then the unique algebra morphism $\Phi : \mathbf{H} \to \mathbf{H}^{\circledast}$ such that $\Phi \circ B = \Gamma^* \circ (\Phi \otimes \Phi)$ is a graded Hopf algebra morphism.*

Proof. Note that Γ is homogeneous of degree -1 and for all $x, y \in \mathbf{H}$,

$$\Gamma(xy) = \Delta(x)\Gamma(y) + \varepsilon(y)\Gamma(x). \tag{20}$$

Then Γ^* is homogeneous of degree 1. Moreover, for all $f, g \in \mathbf{H}^{\circledast}$,

$$\Delta(\Gamma^*(f \otimes g)) = \Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta(f \otimes g). \tag{21}$$

Indeed, if $x, y \in \mathbf{H}$,

$$\begin{array}{rcl} \Delta(\Gamma^*(f \otimes g))(x \otimes y) & = & (f \otimes g)(\Gamma(xy)) \\ & = & (f \otimes g)(\Delta(x)\Gamma(y) + \varepsilon(y)\Gamma(x)) \\ & = & \varepsilon(y)(f \otimes g)(\Gamma(x)) + \Delta(f \otimes g)(\Delta(x) \otimes \Gamma(y)) \\ & = & (\Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta(f \otimes g))\,(x \otimes y). \end{array}$$

With lemma 36 and formula (21), Φ is a Hopf algebra morphism.

Let us prove that it is homogeneous of degree 0. Let F be a forest in $\mathbf{H}$ with n vertices. Then $\Phi(F)$ is homogeneous of degree n. If $n = 0$, then $F = 1$ and $\Phi(F) = 1$ is homogeneous of degree 0. Assume the result is true for all forests with $k < n$ vertices. If F is a tree, $F = B(G \otimes H)$. Then $\Phi(F) = \Gamma^* \circ (\Phi(G) \otimes \Phi(H))$. By the induction hypothesis, $\Phi(G) \otimes \Phi(H)$ is homogeneous of degree $n-1$. As Γ^* is homogeneous of degree 1, $\Phi(F)$ is homogeneous of degree $n-1+1 = n$. If $F = F_1 \ldots F_k$, $k \geq 2$. Then $\Phi(F) = \Phi(F_1) \ldots \Phi(F_k)$ is homogeneous of degree $|F_1| + \ldots + |F_k| = n$ by the induction hypothesis. □

Proof. (of theorem 35) For all $x, y \in \mathbf{H}$, we set $(x, y) = \Phi(x, y)$. We will show that $(-,-)$ is a symmetric Hopf algebra pairing on $\mathbf{H}$ and the bilinear maps $(-,-)$ and $\langle -,- \rangle$ are equal.

First, as Φ is a Hopf algebra morphism, we have for all $x, y, z \in \mathbf{H}$,

$$(1, x) = \Phi(1)(x) = \varepsilon(x) = \varepsilon \circ \Phi(x) = \Phi(x)(1) = (x, 1)\,,$$

$$\begin{array}{rcl} \sum_z \left(x, z^{(1)}\right)\left(y, z^{(2)}\right) & = & \sum_z \Phi(x)(z^{(1)})\Phi(y)(z^{(2)}) \\ & = & (\Phi(x)\Phi(y))(z) \\ & = & \Phi(xy)(z) \\ & = & (xy, z)\,, \\ \sum_x \left(x^{(1)}, y\right)\left(x^{(2)}, z\right) & = & \sum_x \Phi(x^{(1)})(y)\Phi(x^{(2)})(z) \\ & = & \sum_x \Phi(x)^{(1)}(y)\Phi(x)^{(2)}(z) \\ & = & \Phi(x)(yz) \\ & = & (x, yz) \end{array}$$

and

$$(S(x), y) = \Phi(S(x))(y) = S^*(\Phi(x))(y) = \Phi(x)(S(y)) = (x, S(y))\,.$$

Moreover, as Φ is homogeneous of degree 0, if x and y are homogeneous of different degrees, then $\langle x, y \rangle = 0$.

Let $x, y, z \in \mathbf{H}$. Then:

$$\begin{array}{rcl} (B(x \otimes y), z) & = & \Phi \circ B(x \otimes y)(z) \\ & = & \Gamma^* \circ (\Phi(x) \otimes \Phi(y))(z) \\ & = & (\Phi(x) \otimes \Phi(y))(\Gamma(z)) \\ & = & \sum_z \left(x, z_{(1)}\right)\left(y, z_{(2)}\right). \end{array}$$

It remains to prove that $(-,-)$ is symmetric. First we show that for all $x, y, z \in \mathbf{H}$,

$$\sum_x \left(x_{(1)}, y\right)\left(x_{(2)}, z\right) = (x, B(y \otimes z)).$$

We can suppose by bilinearity that x, y and z are three forests. By induction on the degree n of x. If $n = 0$, then $x = 1$ and

$$\sum_x \left(x_{(1)}, y\right)\left(x_{(2)}, z\right) = (0, y)(0, z) = 0,$$
$$(x, B(y \otimes z)) = \varepsilon(B(y \otimes z)) = 0.$$

If $n = 1$, then $x = \bullet$ and $\Gamma(\bullet) = \gamma(\bullet) = 1 \otimes 1$. Then

$$\sum_x \left(x_{(1)}, y\right)\left(x_{(2)}, z\right) = (1, y)(1, z) = \delta_{y,1}\delta_{z,1}$$

$$\begin{array}{rcl} (x, B(y \otimes z)) & = & (B(1 \otimes 1), B(y \otimes z)) \\ & = & \displaystyle\sum_{B(y \otimes z)} \left(1, B(y \otimes z)_{(1)}\right)\left(1, B(y \otimes z)_{(2)}\right) \\ & = & \delta_{B(y \otimes z), \bullet} = \delta_{y,1}\delta_{z,1}. \end{array}$$

Suppose that the result is true for every forest x of degree $< n$. Two cases are possible:

1. If x is a tree, $x = B(x' \otimes x'')$. First, $\Gamma(x) = (-1)^{|x''|} x' \otimes x''$ so

$$\begin{array}{rcl} \displaystyle\sum_x \left(x_{(1)}, y\right)\left(x_{(2)}, z\right) & = & (-1)^{|x''|}\left(x', y\right)\left(x'', z\right) \\ & = & \left\{ \begin{array}{l} 0 \text{ if } deg(x'') \neq deg(z), \\ (-1)^d (x', y)(x'', z) \text{ if } d = deg(x'') = deg(z). \end{array} \right. \end{array}$$

Moreover,

$$\begin{array}{rcl} (x, B(y \otimes z)) & = & \left(B(x' \otimes x''), B(y \otimes z)\right) \\ & = & \displaystyle\sum_{B(y \otimes z)} \left(x', B(y \otimes z)_{(1)}\right)\left(x'', B(y \otimes z)_{(2)}\right) \\ & = & (-1)^{|z|}\left(x', y\right)\left(x'', z\right) \\ & = & \left\{ \begin{array}{l} 0 \text{ if } deg(x'') \neq deg(z), \\ (-1)^d (x', y)(x'', z) \text{ if } d = deg(x'') = deg(z). \end{array} \right. \end{array}$$

2. If x is a forest with at least two trees. Then x can be written $x = x'x''$, with the induction

hypothesis avalaible for x' and x''. Then

$$\begin{array}{rcl}(x, B(y \otimes z)) & = & (x'x'', B(y \otimes z)) \\ & = & \displaystyle\sum_{B(y\otimes z)} \left(x', B(y \otimes z)^{(1)}\right) \left(x'', B(y \otimes z)^{(2)}\right) \\ & = & (x', B(y \otimes z)) \, (x'', 1) + \displaystyle\sum_{y,z} \left(x', y^{(1)} z^{(1)}\right) \left(x'', B(y^{(2)} \otimes z^{(2)})\right) \\ & = & (x', B(y \otimes z)) \, \varepsilon(x'') \\ & & + \displaystyle\sum_{y,z} \left(\sum_{x'} \left(x'^{(1)}, y^{(1)}\right) \left(x'^{(2)}, z^{(1)}\right) \right) \left(\sum_{x''} \left(x''_{(1)}, y^{(2)}\right) \left(x''_{(2)}, z^{(2)}\right) \right) \\ & = & (x', B(y \otimes z)) \, \varepsilon(x'') \\ & & + \displaystyle\sum_{x',x''} \left(\sum_{y} \left(x'^{(1)}, y^{(1)}\right) \left(x''_{(1)}, y^{(2)}\right) \right) \left(\sum_{z} \left(x'^{(2)}, z^{(1)}\right) \left(x''_{(2)}, z^{(2)}\right) \right) \\ & = & \displaystyle\sum_{x'} \left(x'_{(1)}, y\right) \left(x'_{(2)}, z\right) \varepsilon(x'') + \sum_{x',x''} \left(x'^{(1)} x''_{(1)}, y\right) \left(x'^{(2)} x''_{(2)}, z\right) \\ & = & \displaystyle\sum_{x} \langle x_{(1)}, y \rangle \langle x_{(2)}, z \rangle \,, \end{array}$$

by formula (20) for the seventh equality.

So for all $x, y, z \in \mathbf{H}$,

$$\sum_{x} \left(x_{(1)}, y\right) \left(x_{(2)}, z\right) = (x, B(y \otimes z)) \,.$$

Let us prove that $(-,-)$ is symmetric. By induction on the degree n of x. If $n = 0$, then $x = 1$ and

$$(x, y) = (1, y) = \varepsilon(y) = (y, 1) = (y, x) \,.$$

If $deg(x) \geq 1$, two cases are possible:

1. If x is a tree, $x = B(x' \otimes x'')$. Then:

$$\begin{array}{rcl}(x, y) & = & (B(x' \otimes x''), y) \\ & = & \displaystyle\sum_{y} \left(x', y_{(1)}\right) \left(x'', y_{(2)}\right) \\ & = & \displaystyle\sum_{y} \left(y_{(1)}, x'\right) \left(y_{(2)}, x''\right) \\ & = & (y, B(x' \otimes x'')) \\ & = & (y, x) \,. \end{array}$$

2. If x is a forest with at least two trees, x can be written $x = x'x''$ with $deg(x'), deg(x'') < deg(x)$. Then:

$$\begin{array}{rcl}(x, y) & = & (x'x'', y) \\ & = & \displaystyle\sum_{y} \left(x', y^{(1)}\right) \left(x'', y^{(2)}\right) \\ & = & \displaystyle\sum_{y} \left(y^{(1)}, x'\right) \left(y^{(2)}, x''\right) \\ & = & (y, x'x'') \\ & = & (y, x) \,. \end{array}$$

Therefore $(-,-)$ is a symmetric Hopf algebra pairing satisfying the assertions 1-3 of the definition 34. So we have the equality $(-,-) = \langle -,- \rangle$ and this ends the proof. $\square$

Examples. Values of the pairing $\langle -,- \rangle$ for forests of degree ≤ 3:

	$\bullet$
$\bullet$	1

	$\bullet\bullet$	I_l	I_r
$\bullet\bullet$	2	1	1
I_l	1	1	0
I_r	1	0	-1

	$\bullet\bullet\bullet$	$\bullet\,\mathrm{I}_l$	$\mathrm{I}_l\,\bullet$	$\bullet\,\mathrm{I}_r$	$\mathrm{I}_r\,\bullet$	${}_l\mathrm{V}_l$	${}_l\mathrm{V}_r$	${}_r\mathrm{V}_r$	I^l_l	I^r_l	I^l_r	I^r_r
$\bullet\bullet\bullet$	6	3	3	3	3	1	2	1	1	1	1	1
$\bullet\,\mathrm{I}_l$	3	2	2	1	1	1	1	0	1	1	0	0
$\mathrm{I}_l\,\bullet$	3	2	2	1	1	0	1	1	1	0	1	0
$\bullet\,\mathrm{I}_r$	3	1	1	0	0	0	-1	-1	0	0	-1	-1
$\mathrm{I}_r\,\bullet$	3	1	1	0	0	1	1	1	0	-1	0	-1
${}_l\mathrm{V}_l$	1	1	0	0	1	1	0	0	1	1	0	0
${}_l\mathrm{V}_r$	2	1	1	-1	1	0	-1	0	0	0	0	0
${}_r\mathrm{V}_r$	1	0	1	-1	1	0	0	2	0	0	1	1
I^l_l	1	1	1	0	0	1	0	0	1	0	0	0
I^r_l	1	1	0	0	-1	1	0	0	0	-1	0	0
I^l_r	1	0	1	-1	0	0	0	1	0	0	1	0
I^r_r	1	0	0	-1	-1	0	0	1	0	0	0	-1

Question. It is not difficult to see that $\langle -,- \rangle$ is nondegenerate in degree ≤ 3. We conjecture that it is nondegenerate for all degrees.

3.4 Relationship between the coproduct and the bigraft products

Is the coproduct defined in section 3.2 compatible with the bigraft products on the augmentation ideal $\mathbf{M}$ of the Hopf algebra $\mathbf{H}$?

To answer this question, let's first give some results in the case of the right graft algebra $\mathbf{M}_{NCK}$. As $\mathcal{RG}$-algebras are not unitary objects, we need to consider the augmentation ideal of the tensor product of two augmented algebras:

$$\overline{A^+ \otimes B^+} = ((A \oplus \mathbb{K}) \otimes (B \oplus \mathbb{K}))/\mathbb{K}.$$

Let A be a $\mathcal{RG}$-algebra. We extend $\prec: A \otimes A \to A$ to a map $\prec: \overline{A^+ \otimes A^+} \to A$ in the following way : for all $a \in A$, $a \prec 1 = a$ and $1 \prec a = 0$. Moreover, we extend the product of A to a map from $A^+ \otimes A^+$ to A^+ by putting $1 * a = a * 1 = a$ for all $a \in A$ and $1 * 1 = 1$. Note that $1 \prec 1$ is not defined.

Recall the definition of a dipterous algebra (see [LR06]). A (right) dipterous algebra is a $\mathbb{K}$-vector space A equipped with two binary operations denoted by $*$ and $\prec$ satisfying the following relations : for all $x, y, z \in A$,

$$(x * y) * z = x * (y * z), \tag{22}$$
$$(x \prec y) \prec z = x \prec (y * z). \tag{23}$$

Dipterous algebras do not have unit for the product $*$. If A and B are two dipterous algebras, we say that a $\mathbb{K}$-linear map $f : A \to B$ is a dipterous morphism if $f(x * y) = f(x) * f(y)$ and

$f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by *Dipt*-alg the category of dipterous algebras. A $\mathcal{RG}$-algebra is also a dipterous algebra. We get the following canonical functor: $\mathcal{RG}$-alg $\to$ *Dipt*-alg.

Lemma 38 *Let A and B be two $\mathcal{RG}$-algebras. Then $\overline{A^+ \otimes B^+}$ is a dipterous algebra with products defined in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{array}{rcl} (a \otimes b) * (a' \otimes b') &=& (a * a') \otimes (b * b'), \\ (a \otimes b) \prec (a' \otimes b') &=& (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) &=& (a \prec a') \otimes 1. \end{array}$$

Remark. Note that $\overline{A^+ \otimes B^+}$ is not a $\mathcal{RG}$-algebra in general. Indeed, if $A, B \neq \{0\}$, then $(\overline{A^+ \otimes B^+}, *, \prec)$ is a $\mathcal{RG}$-algebra if and only if $\prec: A \otimes A \to A$ and $\prec: B \otimes B \to B$ are zero.

If $\overline{A^+ \otimes B^+}$ is a $\mathcal{RG}$-algebra then for all $a, a' \in A$ and $b, b' \in B$,

$$\begin{array}{rcl} (a \prec a') \otimes b &=& ((a \prec a') \otimes 1) * (1 \otimes b) \\ &=& ((a \otimes 1) \prec (a' \otimes 1)) * (1 \otimes b) \\ &=& (a \otimes 1) \prec (a' \otimes b) \\ &=& (a * a') \otimes (1 \prec b) \\ &=& 0, \end{array}$$

therefore, by taking $b \neq 0$, $a \prec a' = 0$ for all $a, a' \in A$. Moreover,

$$\begin{array}{rcl} a \otimes (b \prec b') &=& (a \otimes b) \prec (1 \otimes b') \\ &=& ((1 \otimes b) * (a \otimes 1)) \prec (1 \otimes b') \\ &=& (1 \otimes b) * (a \otimes (1 \prec b')) \\ &=& 0, \end{array}$$

therefore, by taking $a \neq 0$, $b \prec b' = 0$ for all $b, b' \in B$.

Reciprocally, if $\prec: A \otimes A \to A$ and $\prec: B \otimes B \to B$ are zero, it is clear that $(\overline{A^+ \otimes B^+}, *, \prec)$ is a $\mathcal{RG}$-algebra.

Proof. (of Lemma 38) The associativity of $* : (\overline{A^+ \otimes B^+}) \otimes (\overline{A^+ \otimes B^+}) \to \overline{A^+ \otimes B^+}$ (that is to say the relation (22)) is obvious. We prove (23) : for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{array}{rcl} (a \otimes b) \prec ((a' \otimes b') * (a'' \otimes b'')) &=& (a \otimes b) \prec ((a' * a'') \otimes (b' * b'')) \\ &=& (a * (a' * a'')) \otimes (b \prec (b' * b'')) \\ &=& ((a * a') * a'') \otimes ((b \prec b') \prec b'') \\ &=& ((a * a') \otimes (b \prec b')) \prec (a'' \otimes b'') \\ &=& ((a \otimes b) \prec (a' \otimes b')) \prec (a'' \otimes b''). \end{array}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b'' = 1$ and $b' \in B$,

$$\begin{array}{rcl} (a \otimes 1) \prec ((a' \otimes b') * (a'' \otimes 1)) &=& (a \otimes 1) \prec ((a' * a'') \otimes b') \\ &=& (a * (a' * a'')) \otimes (1 \prec b') \\ &=& 0, \\ ((a \otimes 1) \prec (a' \otimes b')) \prec (a'' \otimes 1) &=& ((a * a') \otimes (1 \prec b')) \prec (a'' \otimes 1) \\ &=& 0. \end{array}$$

If $b = b' = 1$ and $b'' \in B$, then a and a' are not equal to 1 and

$$\begin{array}{rcl} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes b'')) & = & (a \otimes 1) \prec ((a' * a'') \otimes b'') \\ & = & (a * (a' * a'')) \otimes (1 \prec b'') \\ & = & 0, \\ ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes b'') & = & ((a \prec a') \otimes 1) \prec (a'' \otimes b'') \\ & = & ((a * a') \prec a'') \otimes (1 \prec b'') \\ & = & 0. \end{array}$$

Finally if $b = b' = b'' = 1$, then a, a' and a'' are not equal to 1 and

$$\begin{array}{rcl} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes 1)) & = & (a \otimes 1) \prec ((a' * a'') \otimes 1) \\ & = & (a \prec (a' * a'')) \otimes 1 \\ & = & ((a \prec a') \prec a'') \otimes 1 \\ & = & ((a \prec a') \otimes 1) \prec (a'' \otimes 1) \\ & = & ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes 1). \end{array}$$

In all cases, the relation (23) is satisfied and $\overline{A^+ \otimes B^+}$ is a dipterous algebra. □

Proposition 39 *For any tree $T \in \overline{\mathbf{H}_{NCK}}$ and for any forest $F \in \overline{\mathbf{H}_{NCK}}$,*

$$\Delta_{NCK}(T \prec F) = \Delta_{NCK}(T) \prec \Delta_{NCK}(F).$$

In other words, $\Delta_{NCK} : \overline{\mathbf{H}_{NCK}} \to \overline{\mathbf{H}_{NCK} \otimes \mathbf{H}_{NCK}}$ is a dipterous morphism.

Proof. Let T and F are two planar trees of $\overline{\mathbf{H}_{NCK}}$. We note $\Delta_{NCK}(T) = T \otimes 1 + 1 \otimes T + \sum_T T^{(1)} \otimes T^{(2)}$ and $\Delta_{NCK}(F) = F \otimes 1 + 1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}$. Then

$$\begin{array}{rl} & \Delta_{NCK}(T) \prec \Delta_{NCK}(F) \\ = & \left(T \otimes 1 + 1 \otimes T + \sum_T T^{(1)} \otimes T^{(2)}\right) \prec \left(F \otimes 1 + 1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}\right) \\ = & (T \prec F) \otimes 1 + T \otimes F + \sum_T T^{(1)} F \otimes T^{(2)} + 1 \otimes (T \prec F) \\ & + \sum_T T^{(1)} \otimes (T^{(2)} \prec F) + \sum_F F^{(1)} \otimes (T \prec F^{(2)}) \\ & + \sum_{T,F} T^{(1)} F^{(1)} \otimes (T^{(2)} \prec F^{(2)}) \\ = & \Delta_{NCK}(T \prec F). \end{array}$$

If $F = F_1 \dots F_n \in \overline{\mathbf{H}_{NCK}}$ is a forest and T is again a tree of $\overline{\mathbf{H}_{NCK}}$,

$$\begin{array}{rl} & \Delta_{NCK}(T) \prec \Delta_{NCK}(F) \\ = & \Delta_{NCK}(T) \prec (\Delta_{NCK}(F_1) \dots \Delta_{NCK}(F_n)) \\ = & (\dots((\Delta_{NCK}(T) \prec \Delta_{NCK}(F_1)) \prec \Delta_{NCK}(F_2)) \dots \prec \Delta_{NCK}(F_{n-1})) \prec \Delta_{NCK}(F_n) \\ = & \Delta_{NCK}((\dots((T \prec F_1) \prec F_2) \dots \prec F_{n-1}) \prec F_n) \\ = & \Delta_{NCK}(T \prec F), \end{array}$$

as $\overline{\mathbf{H}_{NCK} \otimes \mathbf{H}_{NCK}}$ is a dipterous algebra, for the second equality. □

In fact, the right graft product is not fully compatible with the given coproduct. Indeed, as $\overline{\mathbf{H}_{NCK} \otimes \mathbf{H}_{NCK}}$ is not a $\mathcal{RG}$-algebra, if $T \in \overline{\mathbf{H}_{NCK}}$ is a forest, $\Delta_{NCK}(T \prec F) \neq \Delta_{NCK}(T) \prec \Delta_{NCK}(F)$ in general. For example,

$$\begin{array}{rcl} \Delta_{NCK}((\bullet\bullet) \prec \bullet) & = & \Delta_{NCK}(\bullet \overset{\bullet}{\underset{\bullet}{|}}) \\ & = & \bullet \overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + 1 \otimes \bullet \overset{\bullet}{\underset{\bullet}{|}} + \overset{\bullet}{\underset{\bullet}{|}} \otimes \bullet + \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet\bullet \otimes \bullet + \bullet \otimes \bullet\bullet \\ \Delta_{NCK}(\bullet\bullet) \prec \Delta_{NCK}(\bullet) & = & (\bullet\bullet \otimes 1 + 1 \otimes \bullet\bullet + 2 \bullet \otimes \bullet) \prec (\bullet \otimes 1 + 1 \otimes \bullet) \\ & = & \bullet \overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + 1 \otimes \bullet \overset{\bullet}{\underset{\bullet}{|}} + \bullet \otimes \bullet\bullet + 2 \bullet\bullet \otimes \bullet + 2 \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}}. \end{array}$$

We now focus on the case of bigraft algebras. We will encounter the same difficulties. As bigraft algebras are not objects with unit, we consider again the extended tensor product $\overline{A^+ \otimes B^+}$ for A, B two $\mathcal{BG}$-algebras. If A is a bigraft algebra, we extend $\succ, \prec \colon A \otimes A \to A$ into maps $\succ, \prec \colon \overline{A^+ \otimes A^+} \to A$ in the following way : for all $a \in A$,

$$a \succ 1 = 0, \qquad a \prec 1 = a, \qquad 1 \succ a = a, \qquad 1 \prec a = 0.$$

Moreover, we extend the product of A into a map from $A^+ \otimes A^+$ to A^+ by putting $1 * a = a * 1 = a$ for all $a \in A$ and $1 * 1 = 1$. Note that relations (7) are now satisfied on $\overline{A^+ \otimes A^+ \otimes A^+}$.

We recall (see [Ler03]) that a pre-dendriform algebra is a $\mathbb{K}$-vector space A equipped with three binary operations denoted by $*$, $\succ$ and $\prec$ satisfying the four relations : for all $x, y, z \in A$,

$$\begin{array}{rcl} (x * y) * z & = & x * (y * z), \\ (x * y) \succ y & = & x \succ (y \succ z), \\ (x \prec y) \prec z & = & x \prec (y * z), \\ (x \succ y) \prec z & = & x \succ (y \prec z). \end{array}$$

In other words, $(A, *, \succ)$ and $(A, *, \prec)$ are dipterous algebras with the entanglement relation $(x \succ y) \prec z = x \succ (y \prec z)$.

Pre-dendriform algebras do not have unit for the product $*$. If A and B are two dipterous algebras, a $\mathbb{K}$-linear map $f : A \to B$ is a pre-dendriform morphism if $f(x * y) = f(x) * f(y)$, $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$. Let us denote by $PreDend$-alg the category of pre-dendriform algebras. A $\mathcal{BG}$-algebra is also a pre-dendriform algebra. We get the following canonical functor: $\mathcal{BG}$-alg $\to PreDend$-alg.

Lemma 40 *Let A and B be two bigraft algebras. Then $\overline{A^+ \otimes B^+}$ is given a structure of pre-dendriform algebra in the following way : for all $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{array}{rcl} (a \otimes b) * (a' \otimes b') & = & (a * a') \otimes (b * b'), \\ (a \otimes b) \succ (a' \otimes b') & = & (a * a') \otimes (b \succ b'), \ \textit{if } b \textit{ or } b' \in B, \\ (a \otimes 1) \succ (a' \otimes 1) & = & (a \succ a') \otimes 1, \\ (a \otimes b) \prec (a' \otimes b') & = & (a * a') \otimes (b \prec b'), \ \textit{if } b \textit{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) & = & (a \prec a') \otimes 1. \end{array}$$

Remark. Note that $\overline{A^+ \otimes B^+}$ is not a $\mathcal{BG}$-algebra in general.

Proof. With lemma 38, $(\overline{A^+ \otimes B^+}, *, \prec)$ is a right dipterous algebra. In the same way, $(\overline{A^+ \otimes B^+}, *, \succ)$ is a left dipterous algebra. It remains to show the entanglement relation: for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{array}{rcl} ((a \otimes b) \succ (a' \otimes b')) \prec (a'' \otimes b'') & = & ((a * a') * a'') \otimes ((b \succ b') \prec b'') \\ & = & (a * (a' * a'')) \otimes (b \succ (b' \prec b'')) \\ & = & (a \otimes b) \succ ((a' \otimes b') \prec (a'' \otimes b'')). \end{array}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b' = 1$ and $b'' \in B$,

$$\begin{array}{rcl} ((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes b'') &=& ((a \succ a') * a'') \otimes (1 \prec b'') \\ &=& 0, \\ (a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes b'')) &=& (a \otimes 1) \succ ((a' * a'') \otimes (1 \prec b'')) \\ &=& 0. \end{array}$$

If $b' = b'' = 1$ and $b \in B$, it is the same calculation as previously. Finally if $b = b' = b'' = 1$,

$$\begin{array}{rcl} ((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes 1) &=& ((a \succ a') \prec a'') \otimes 1 \\ &=& (a \succ (a' \prec a'')) \otimes 1 \\ &=& (a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes 1)). \end{array}$$

□

Theorem 41 *For all forests $F, G \in \overline{\mathbf{H}}$ and for all tree $T \in \overline{\mathbf{H}}$,*

$$\Delta(F \succ T \prec G) = \Delta(F) \succ \Delta(T) \prec \Delta(G). \tag{24}$$

In other words, $\Delta : \overline{\mathbf{H}} \to \overline{\mathbf{H} \otimes \mathbf{H}}$ is a pre-dendriform morphism.

Proof. With proposition 39, we have $\Delta(T \prec G) = \Delta(T) \prec \Delta(G)$. Moreover,

$$\begin{array}{rcl} \Delta(F \succ T) &=& \Delta((F^\dagger)^\dagger \succ (T^\dagger)^\dagger) \\ &=& (\dagger \otimes \dagger) \circ \Delta(T^\dagger \prec F^\dagger) \\ &=& (\dagger \otimes \dagger) \circ (\Delta(T^\dagger) \prec \Delta(F^\dagger)) \\ &=& (\dagger \otimes \dagger) \circ (((\dagger \otimes \dagger) \circ \Delta(T)) \prec ((\dagger \otimes \dagger) \circ \Delta(F))) \\ &=& (\dagger \otimes \dagger) \circ (\dagger \otimes \dagger) \circ (\Delta(F) \succ \Delta(T)) \\ &=& \Delta(F) \succ \Delta(T), \end{array}$$

using propositions 33 and 14. As the entanglement relation is satisfied in $\overline{\mathbf{H}}$ and $\overline{\mathbf{H} \otimes \mathbf{H}}$, any parenthesizing of (24) gives the same relation. □

Remark. As $\overline{\mathbf{H} \otimes \mathbf{H}}$ is not a $\mathcal{BG}$-algebra, the relation (24) is not true if T is a forest in general.

4 A good triple of operads $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$

The aim of this section is to prove an analogue of Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems in the case of bigraft algebras.

4.1 Good triple of operads and rigidity theorem for the right graft algebras

In this subsection, we recall some results on generalized bialgebra and good triple of operads (see [Lod08]).

Let $\mathcal{A}$ and $\mathcal{C}$ be two algebraic operads. A generalized bialgebra associated with $\mathcal{A}$ and $\mathcal{C}$, or $\mathcal{C}^c$-$\mathcal{A}$-bialgebra, is a $\mathbb{K}$-vector space H which is an $\mathcal{A}$-algebra, a $\mathcal{C}$-coalgebra, and such that the operations of $\mathcal{A}$ and the cooperations of $\mathcal{C}$ acting on H satisfy some compatibility relations.

Suppose that the following two hypothesis are fulfilled:

- There is a distributive compatibility relation for any pair (δ, μ) where μ is an operation and δ is a cooperation.
- The free $\mathcal{A}$-algebra $\mathcal{A}(V)$ over a $\mathbb{K}$-vector space V is equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure.

Then it determines an operad $\mathcal{P} := Prim_{\mathcal{C}}\mathcal{A}$ and a functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$. The operad $\mathcal{P}$ is the largest suboperad of $\mathcal{A}$ such that any $\mathcal{P}$-operation applied on primitive elements gives a primitive element. For any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H the inclusion $Prim(H) \hookrightarrow H$ becomes a morphism of $\mathcal{P}$-algebras. J.-L. Loday calls this whole structure a *triple of operads*, and denotes it by $(\mathcal{C}, \mathcal{A}, \mathcal{P})$.

The functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$ is a forgetful functor in the sense that the composition $\mathcal{A}\text{-alg} \xrightarrow{F} \mathcal{P}\text{-alg} \to \text{Vect}$ is the forgetful functor $\mathcal{A}\text{-alg} \to \text{Vect}$. This forgetful functor has a left adjoint denoted by $U : \mathcal{P}\text{-alg} \to \mathcal{A}\text{-alg}$ and called the *universal enveloping algebra functor.*

A triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a *good triple of operads* if $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ satisfies the following structure theorem: for any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H the following are equivalent:
1. The $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H is connected.
2. There is an isomorphism of connected coalgebras $H \cong \mathcal{C}^c(Prim(H))$.
3. There is an isomorphism of $\mathcal{C}^c$-$\mathcal{A}$-bialgebras $H \cong U(Prim(H))$.

We now give a rigidity theorem from [Foi10] concerning right graft algebras. Consider the deconcatenation coproduct $\Delta_{\mathcal{A}ss}$ on $\mathbf{H}_{NCK}$: for any forest $F \in \mathbf{H}_{NCK}$,

$$\Delta_{\mathcal{A}ss}(F) = \sum_{F_1, F_2 \in \mathbf{H}_{NCK}, F_1 F_2 = F} F_1 \otimes F_2.$$

Notation. Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, non zero, such that $\Delta(1) = 1 \otimes 1$. We then define the noncounitary coproduct:

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} Ker(\varepsilon) & \to & Ker(\varepsilon) \otimes Ker(\varepsilon), \\ a & \mapsto & \Delta(a) - a \otimes 1 - 1 \otimes a. \end{array} \right.$$

Let $\tilde{\Delta}_{\mathcal{A}ss}$ be the coproduct on $\mathbf{M}_{NCK}$ obtained from $\Delta_{\mathcal{A}ss}$ by substracting its primitive part. We now have two products m and $\succ$ and one coproduct $\tilde{\Delta}_{\mathcal{A}ss}$ on $\mathbf{M}_{NCK}$.

Note that if A and B are two $\mathcal{RG}$-algebras then $\overline{A^+ \otimes B^+}$ is a dipterous algebra (lemma 38). Recall that (see [Foi10]) an infinitesimal right graft bialgebra is a family $(A, m, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ where A is a $\mathbb{K}$-vector space, $m, \prec$: $\overline{A^+ \otimes A^+} \to A$ and $\tilde{\Delta}_{\mathcal{A}ss} : A \to \overline{A^+ \otimes A^+}$ are $\mathbb{K}$-linear maps, with the following compatibilities :

1. $(A, m, \prec)$ is a right graft algebra.
2. For all $x, y \in A$:

$$\left\{ \begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(xy) & = & (x \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array} \right. \tag{25}$$

It is proved in [Foi10] that $(\mathbf{M}_{NCK}, m, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ is an infinitesimal right graft bialgebra. In particular, with the first equality of (25), $(\mathbf{M}_{NCK}, \tilde{\Delta}_{\mathcal{A}ss})$ is an infinitesimal bialgebra (see [LR06]). If A is an infinitesimal right graft bialgebra, we note $Prim(A) = Ker(\tilde{\Delta}_{\mathcal{A}ss})$.

Let us recall that a magmatic algebra is a $\mathbb{K}$-vector space A equipped with a binary operation $\bullet$, without any relation. We do not suppose that magmatic algebras have units. The operad associated is denoted by $\mathcal{M}ag$.

L. Foissy prove in [Foi10] that for any infinitesimal right graft bialgebra, its primitive part is a $\mathcal{M}ag$-algebra. In particular, $Prim(\mathbf{M}_{NCK}) = \mathbb{K}(\mathbb{T}_{NCK})$ equipped with the product $\prec$ is a $\mathcal{M}ag$-algebra and it is the free $\mathcal{M}ag$-algebra generated by $\bullet$.

Then we have the following important result (see [Foi10]): The triple $(\mathcal{A}ss, \mathcal{RG}, \mathcal{M}ag)$ is a good triple of operads.

4.2 Infinitesimal bigraft bialgebras

In section 3.4, we gave the relationship between the coproduct Δ defined in section 3.2 and the bigraft products (see theorem 41). These relationships do not permit to define a notion of bigraft bialgebra.

Consider an another coproduct on $\mathbf{H}$, the deconcatenation coproduct $\Delta_{\mathcal{A}ss}$: for all $F \in \mathbb{F}$,

$$\Delta_{\mathcal{A}ss}(F) = \sum_{F_1, F_2 \in \mathbb{F}, F_1 F_2 = F} F_1 \otimes F_2.$$

We will show that, with this coproduct, we have good relationship with the bigraft products and we can define the notion of bigraft bialgebra on planar decorated trees.

We give in the following proposition another definition of $\succ$ and $\prec$, to be compared with lemma 40 (and we will use this definition in the following) :

Definition 42 *Let A and B be two bigraft algebras. Then we define three binary operations denoted by $\succ, \prec$ and $*$ on $\overline{A^+ \otimes B^+}$ in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{array}{rcl}
(a \otimes b) * (a' \otimes b') & = & (a * a') \otimes (b * b'), \\
(a \otimes b) \succ (a' \otimes b') & = & (a \succ a') \otimes (b * b'), \text{ if } a \text{ or } a' \in A, \\
(1 \otimes b) \succ (1 \otimes b') & = & 1 \otimes (b \succ b'), \\
(a \otimes b) \prec (a' \otimes b') & = & (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\
(a \otimes 1) \prec (a' \otimes 1) & = & (a \prec a') \otimes 1.
\end{array}$$

Remark. With lemma 38, $(\overline{A^+ \otimes B^+}, *, \prec)$ is a right dipterous algebra. Applying the same reasoning to $(\overline{A^+ \otimes B^+}, *, \succ)$, we prove that this is a left dipterous algebra. Moreover, the entanglement relation is not true in general : for $a, a'' \in A$ and $b, b', b'' \in B$,

$$\begin{array}{rcl}
((a \otimes b) \succ (1 \otimes b')) \prec (a'' \otimes b'') & = & ((a \succ 1) \otimes (b * b')) \prec (a'' \otimes b'') = 0 \\
(a \otimes b) \succ ((1 \otimes b') \prec (a'' \otimes b'')) & = & (a \succ a'') \otimes (b * (b' \prec b'')).
\end{array}$$

We now have three products, namely m, $\succ$ and $\prec$ and one coproduct, namely $\tilde{\Delta}_{\mathcal{A}ss}$, on $\mathbf{M}$, obtained from $\Delta_{\mathcal{A}ss}$ by substracting its primitive parts. We introduce the definition of infinitesimal bigraft bialgebra:

Definition 43 *An infinitesimal bigraft bialgebra is a family $(A, m, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ where $m, \succ, \prec$: $\overline{A^+ \otimes A^+} \to A$, $\tilde{\Delta}_{\mathcal{A}ss} : A \to \overline{A^+ \otimes A^+}$, with the following compatibilities :*

1. *$(A, m, \succ, \prec)$ is a bigraft algebra.*
2. *For all $x, y \in A$:*

$$\left\{ \begin{array}{rcl}
\tilde{\Delta}_{\mathcal{A}ss}(xy) & = & (x \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x)(1 \otimes y) + x \otimes y, \\
\tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\
\tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y).
\end{array} \right. \quad (26)$$

Then the following properties illustrate the compatibility relationships:

Proposition 44 $(\mathbf{M}, m, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ *is an infinitesimal bigraft bialgebra. In particular, for all* $x, y \in \mathbf{M}$,

$$\left\{ \begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(xy) & = & (x \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array} \right. \tag{27}$$

Proof. With the proposition 11, $(\mathbf{M}, m, \succ, \prec)$ is a bigraft algebra. It remains to prove the formulas (27). We can restrict ourselves to $F, G \in \mathbb{F} \setminus \{1\}$. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$ where the F_i's and the G_i's are trees and $G_1 = B(G_1^1 \otimes G_1^2)$. Hence :

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(FG) & = & \displaystyle\sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = FG} H_1 \otimes H_2 \\ & = & \displaystyle\sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = G} FH_1 \otimes H_2 + \sum_{H_1, H_2 \in \mathbb{F} \setminus \{1\}, H_1 H_2 = F} H_1 \otimes H_2 G + F \otimes G \\ & = & (F \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(G) + \tilde{\Delta}_{\mathcal{A}ss}(F)(1 \otimes G) + F \otimes G, \end{array}$$

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(F \succ G) & = & \tilde{\Delta}_{\mathcal{A}ss}(B(FG_1^1 \otimes G_1^2)G_2 \dots G_m) \\ & = & B(FG_1^1 \otimes G_1^2) \otimes G_2 \dots G_m + \displaystyle\sum_{i=2}^{m-1} B(FG_1^1 \otimes G_1^2)G_2 \dots G_i \otimes G_{i+1} \dots G_m \\ & = & F \succ G_1 \otimes G_2 \dots G_m + \displaystyle\sum_{i=2}^{m-1} F \succ G_1 G_2 \dots G_i \otimes G_{i+1} \dots G_m \\ & = & (F \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(G). \end{array}$$

Remark that $\Delta_{\mathcal{A}ss}(F^\dagger) = \tau \circ (\dagger \otimes \dagger) \circ \Delta_{\mathcal{A}ss}(F)$ for all $F \in \mathbb{F}$. So we deduce the third relation of (27) from the second one in this way :

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(F \prec G) & = & \tilde{\Delta}_{\mathcal{A}ss}((G^\dagger \succ F^\dagger)^\dagger) \\ & = & \tau \circ (\dagger \otimes \dagger) \circ \tilde{\Delta}_{\mathcal{A}ss}(G^\dagger \succ F^\dagger) \\ & = & \tau \circ (\dagger \otimes \dagger)((G^\dagger \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger)) \\ & = & \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger) \prec (1 \otimes G). \end{array}$$

□

Definition 45 *If* A *is an infinitesimal bigraft bialgebra, we note* $Prim(A) = Ker(\tilde{\Delta}_{\mathcal{A}ss})$. *In the infinitesimal bigraft bialgebra* $\mathbf{M}$, $Prim(\mathbf{M}) = \mathbb{K}(\mathbb{T})$ *and we denote by* $\mathbf{P}$ *the primitive part of* $\mathbf{M}$.

Recall the definition of a $\mathcal{L}$-algebra introduced by P. Leroux in [Ler08] :

Definition 46 *A* $\mathcal{L}$*-algebra is a* $\mathbb{K}$*-vector space* A *equipped with two binary operations* $\succ, \prec$: $A \otimes A \to A$ *verifying the entanglement relation:*

$$(x \succ y) \prec z = x \succ (y \prec z),$$

for all $x, y, z \in A$.

The operad $\mathcal{L}$ is binary, quadratic, regular and set-theoretic. We denote by $\widetilde{\mathcal{L}}$ the nonsymmetric operad associated with the regular operad $\mathcal{L}$. We do not suppose that $\mathcal{L}$-algebras have unit for $\succ$ or $\prec$. If A and B are two $\mathcal{L}$-algebras, a $\mathcal{L}$-morphism from A to B is a $\mathbb{K}$-linear map $f : A \to B$ such that $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by $\mathcal{L}$-alg the category of $\mathcal{L}$-algebras.

Proposition 47 *For any infinitesimal bigraft bialgebra, its primitive part is a $\mathcal{L}$-algebra.*

Proof. Let A be an infinitesimal bigraft bialgebra. We put $x, y \in Prim(A)$. Then, with (26),

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y) = 0, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y) = 0. \end{array}$$

Therefore, $x \succ y, x \prec y \in Prim(A)$. As we always have the relation $(x \succ y) \prec z = x \succ (y \prec z)$ for all $x, y, z \in Prim(A)$, $Prim(A)$ is a $\mathcal{L}$-algebra. □

In particular, $(\mathbf{P}, \succ, \prec)$ is a $\mathcal{L}$-algebra. We have even more than this:

Theorem 48 $(\mathbf{P}, \succ, \prec)$ *is the free $\mathcal{L}$-algebra generated by* $\bullet$.

Proof. Let A be a $\mathcal{L}$-algebra and $a \in A$. Let us prove that there exists a unique morphism of $\mathcal{L}$-algebras $\phi : \mathbf{P} \to A$ such that $\phi(\bullet) = a$. We define $\phi(F)$ for any nonempty tree $F \in \mathbf{P}$ inductively on the degree of F by:

$$\left\{ \begin{array}{rcl} \phi(\bullet) & = & a, \\ \phi(F) & = & (\ldots((\phi(F_1^1) \succ (\ldots(\phi(F_{p-1}^1) \succ (\phi(F_p^1) \succ a))\ldots)) \prec \phi(F_1^2))\ldots) \prec \phi(F_q^2) \\ & & \text{if } F = B(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2) \text{ with the } F_i^1\text{'s and the } F_j^2\text{'s} \in \mathbb{T}. \end{array} \right.$$

This map is linearly extended into a map $\phi : \mathbf{P} \to A$. Let us show that it is a morphism of $\mathcal{L}$-algebras, that is to say $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$ for all $F, G \in \mathbb{T} \setminus \{1\}$. Note $F = B(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2)$, $G = B(G_1^1 \ldots G_r^1 \otimes G_1^2 \ldots G_s^2)$ with $F_1^1, \ldots, F_p^1, F_1^2, \ldots, F_q^2$ and $G_1^1, \ldots, G_r^1, G_1^2, \ldots, G_s^2$ in $\mathbb{T}$. Then:

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,

$$\begin{array}{rcl} \phi(F \succ G) & = & \phi(B(FG_1^1 \ldots G_r^1 \otimes G_1^2 \ldots G_s^2)) \\ & = & (\ldots((\phi(F) \succ (\phi(G_1^1) \succ (\ldots(\phi(G_r^1) \succ a)\ldots))) \prec \phi(G_1^2))\ldots) \prec \phi(G_s^2) \\ & = & \phi(F) \succ ((\ldots((\phi(G_1^1) \succ (\ldots(\phi(G_r^1) \succ a)\ldots)) \prec \phi(G_1^2))\ldots) \prec \phi(G_s^2)) \\ & = & \phi(F) \succ \phi(G). \end{array}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{array}{rcl} \phi(F \prec G) & = & \phi(B(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2 G)) \\ & = & ((\ldots((\phi(F_1^1) \succ (\ldots(\phi(F_p^1) \succ a)\ldots)) \prec \phi(F_1^2))\ldots) \prec \phi(F_q^2)) \prec \phi(G) \\ & = & \phi(F) \prec \phi(G). \end{array}$$

So ϕ is a morphism of $\mathcal{L}$-algebras.

Let $\phi' : \mathbf{P} \to A$ be another morphism of $\mathcal{L}$-algebras such that $\phi'(\bullet) = a$. For any forest $F_1^1 \ldots F_p^1$ and $F_1^2 \ldots F_q^2 \in \mathbb{F}$,

$$\begin{array}{rl} & \phi'(B(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2)) \\ = & \phi'((\ldots((F_1^1 \succ (\ldots(F_{p-1}^1 \succ (F_p^1 \succ \bullet))\ldots)) \prec F_1^2)\ldots) \prec F_q^2) \\ = & (\ldots((\phi'(F_1^1) \succ (\ldots(\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ \phi'(\bullet)))\ldots)) \prec \phi'(F_1^2))\ldots) \prec \phi'(F_q^2) \\ = & (\ldots((\phi'(F_1^1) \succ (\ldots(\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ a))\ldots)) \prec \phi'(F_1^2))\ldots) \prec \phi'(F_q^2). \end{array}$$

So $\phi = \phi'$. □

Remark. We deduce from theorem 48 that $dim\left(\widetilde{\mathcal{L}}(n)\right) = t_n^{\mathbf{H}}$ for all $n \in \mathbb{N}^*$ (with the same reasoning as in corollary 16). We find again the result already given in [Ler08].

4.3 Rigidity theorem for infinitesimal bigraft bialgebras

The functor $(-)_{\mathcal{L}} : \{\mathcal{BG}-\text{alg}\} \to \{\mathcal{L}-\text{alg}\}$ associates to a $\mathcal{BG}$-algebra $(A, m, \succ, \prec)$ the $\mathcal{L}$-algebra $(A, \succ, \prec)$. We define the adjoint functor $U_{\mathcal{BG}}(-) : \{\mathcal{L} - \text{alg}\} \to \{\mathcal{BG} - \text{alg}\}$, called the universal enveloping bigraft algebra functor, as follows:

Definition 49 *The universal enveloping bigraft algebra of a $\mathcal{L}$-algebra $(A, \succ, \prec)$, denoted by $U_{\mathcal{BG}}(A)$, is the augmentation ideal $\overline{T}(A)$ of the tensor algebra $T(A)$ over the $\mathbb{K}$-vector space A equipped with two operations also denoted by $\succ$ and $\prec$, and defined by : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \le i \le p$ and $1 \le j \le q$,*

$$\begin{array}{rcl} (a_1 \ldots a_p) \succ (b_1 \ldots b_q) & := & (a_1 \succ (\ldots (a_{p-1} \succ (a_p \succ b_1)) \ldots)) b_2 \ldots b_q, \\ (a_1 \ldots a_p) \prec (b_1 \ldots b_q) & := & a_1 \ldots a_{p-1}((\ldots ((a_p \prec b_1) \prec b_2) \ldots) \prec b_q). \end{array} \quad (28)$$

Then $(\overline{T}(A), m, \succ, \prec)$ is a nonunitary bigraft algebra, where m is the concatenation.

We denote by $\mathcal{L}(V)$ the free $\mathcal{L}$-algebra over a $\mathbb{K}$-vector space V. The functor $\mathcal{L}(-)$ is the left adjoint to the forgetful functor from $\mathcal{L}$-algebras to vector spaces. Because the operad $\mathcal{L}$ is regular, we get the following result :

Proposition 50 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{L}$-algebra on V is*

$$\mathcal{L}(V) = \bigoplus_{n \ge 1} \mathbb{K}\,(\mathbb{T}(n)) \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{T}(n)$, $G \in \mathbb{T}(m)$, $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \ldots \otimes w_m \in V^{\otimes m}$,

$$\begin{array}{rcl} (F \otimes v_1 \otimes \ldots \otimes v_n) \succ (G \otimes w_1 \otimes \ldots \otimes w_m) & = & (F \succ G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m), \\ (F \otimes v_1 \otimes \ldots \otimes v_n) \prec (G \otimes w_1 \otimes \ldots \otimes w_m) & = & (F \prec G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m). \end{array}$$

Proposition 51 *The universal enveloping bigraft algebra of the free $\mathcal{L}$-algebra is canonically isomorphic to the free bigraft algebra :*

$$U_{\mathcal{BG}}(\mathcal{L}(V)) \cong \mathcal{BG}(V).$$

Proof. Let us prove that the functor $U_{\mathcal{BG}} : \{\mathcal{L} - \text{alg}\} \to \{\mathcal{BG} - \text{alg}\}$ is the left adjoint to $(-)_{\mathcal{L}} : \{\mathcal{BG} - \text{alg}\} \to \{\mathcal{L} - \text{alg}\}$.

We put A a $\mathcal{L}$-algebra and B a $\mathcal{BG}$-algebra. Let $f : A \to (B)_{\mathcal{L}}$ be a morphism of $\mathcal{L}$-algebras. It determines uniquely a morphism of algebras $\tilde{f} : \overline{T}(A) \to B$ because (B, m) is an associative algebra. We endow $\overline{T}(A)$ with a $\mathcal{BG}$-algebra structure defined by (28). Then $\tilde{f} : U_{\mathcal{BG}}(A) \to B$ is a morphism of $\mathcal{BG}$-algebras : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \le i \le p$ and $1 \le j \le q$,

$$\begin{array}{rcl} \tilde{f}((a_1 \ldots a_p) \succ (b_1 \ldots b_q)) & = & \tilde{f}((a_1 \succ (\ldots (a_{p-1} \succ (a_p \succ b_1)) \ldots)) b_2 \ldots b_q) \\ & = & (\tilde{f}(a_1) \succ (\ldots (\tilde{f}(a_{p-1}) \succ (\tilde{f}(a_p) \succ \tilde{f}(b_1))) \ldots)) \tilde{f}(b_2) \ldots \tilde{f}(b_q) \\ & = & (\tilde{f}(a_1) \ldots \tilde{f}(a_p)) \succ (\tilde{f}(b_1) \ldots \tilde{f}(b_q)) \\ & = & \tilde{f}(a_1 \ldots a_p) \succ \tilde{f}(b_1 \ldots b_q), \end{array}$$

and

$$\begin{array}{rcl} \tilde{f}((a_1 \ldots a_p) \prec (b_1 \ldots b_q)) & = & \tilde{f}(a_1 \ldots a_{p-1}((\ldots ((a_p \prec b_1) \prec b_2) \ldots) \prec b_q)) \\ & = & \tilde{f}(a_1) \ldots \tilde{f}(a_{p-1})((\ldots ((\tilde{f}(a_p) \prec \tilde{f}(b_1)) \prec \tilde{f}(b_2)) \ldots) \prec \tilde{f}(b_q)) \\ & = & (\tilde{f}(a_1) \ldots \tilde{f}(a_p)) \prec (\tilde{f}(b_1) \ldots \tilde{f}(b_q)) \\ & = & \tilde{f}(a_1 \ldots a_p) \prec \tilde{f}(b_1 \ldots b_q). \end{array}$$

On the other hand, let $g : U_{\mathcal{BG}}(A) \to B$ be a morphism of $\mathcal{BG}$-algebras. From the construction of $U_{\mathcal{BG}}(A)$ it follows that the map $A \to U_{\mathcal{BG}}(A)$ is a $\mathcal{L}$-algebra morphism. Hence the composition $\tilde{g}$ with g gives a $\mathcal{L}$-algebra morphism $A \to B$.

These two constructions are inverse of each other, and therefore $U_{\mathcal{BG}}$ is the left adjoint to $(-)_{\mathcal{L}}$.

As $U_{\mathcal{BG}}$ is left adjoint to $(-)_{\mathcal{L}}$ and $\mathcal{L}(-)$ is left adjoint to the forgetful functor, the composite is the left adjoint to the forgetful functor from $\mathcal{BG}$-algebras to vector spaces. Hence it is the functor $\mathcal{BG}(-)$. □

Theorem 52 *For any infinitesimal bigraft bialgebra A over a field $\mathbb{K}$, the following are equivalent:*

1. *A is a connected infinitesimal bigraft bialgebra,*
2. *A is cofree among the connected coalgebras,*
3. *A is isomorphic to $U_{\mathcal{BG}}(Prim(A))$ as an infinitesimal bigraft bialgebra.*

Proof. We prove the following implications 1. ⇒ 2. ⇒ 3. ⇒ 1.
1. ⇒ 2. Suppose that A is a connected infinitesimal bigraft bialgebra. Then A is isomorphic to $(\overline{T}(Prim(A)), m, \Delta)$ as an infinitesimal bialgebra, where $\overline{T}(Prim(A))$ is the augmentation ideal of the tensor algebra over $Prim(A)$, m is the concatenation and Δ is the deconcatenation (see [LR06] for a proof). Therefore A is cofree.
2. ⇒ 3. If A is cofree, then it is isomorphic as an infinitesimal bialgebra to $(\overline{T}(Prim(A)), m, \Delta)$ and $Prim(A)$ is a $\mathcal{L}$-algebra with proposition 47. $\overline{T}(Prim(A))$ is a $\mathcal{BG}$-algebra with the two operations $\succ$ and $\prec$ defined as in (28) and this is exactly $U_{\mathcal{BG}}(Prim(A))$. So A is isomorphic as an infinitesimal bialgebra to $U_{\mathcal{BG}}(Prim(A))$ and it is a $\mathcal{BG}$-morphism by using (7) and (28).
3. ⇒ 1. By construction, $U_{\mathcal{BG}}(Prim(A))$ is isomorphic to $\overline{T}(Prim(A))$ as a bialgebra. Therefore A is isomorphic to $\overline{T}(Prim(A))$ as a bialgebra. As $\overline{T}(Prim(A))$ is connected, A is connected. □

We deduce the following theorem:

Theorem 53 *The triple $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ is a good triple of operads.*

Remark. Note that if A is an infinitesimal bigraft bialgebra, then $(A, m, \tilde{\Delta}_{\mathcal{A}ss})$ is a nonunitary infinitesimal bialgebra. Hence, if $(\mathbb{K} \oplus A, m, \Delta_{\mathcal{A}ss})$ has an antipode S, then $-S$ is an eulerian idempotent for A and we have :

$$S(a) = \begin{cases} -a \text{ if } a \in Prim(A), \\ 0 \text{ if } a \in A^2. \end{cases}$$

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