Preordered forests, packed words and contraction algebras

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Abstract. We introduce the notions of preordered and heap-preordered forests, generalizing the construction of ordered and heap-ordered forests. We prove that the algebras of preordered and heap-preordered forests are Hopf for the cut coproduct, and we construct a Hopf morphism to the Hopf algebra of packed words. Moreover, we define another coproduct on the preordered forests given by the contraction of edges. Finally, we give a combinatorial description of morphims defined on Hopf algebras of forests with values in the Hopf algebras of shuffes or quasi-shuffles.

Résumé. Nous introduisons les notions de forêts préordonnées et préordonnées en tas, généralisant les constructions des forêts ordonnées et ordonnées en tas. On démontre que les algèbres des forêts préordonnées et préordonnées en tas sont des algèbres de Hopf pour le coproduit de coupes et on construit un morphisme d'algèbres de Hopf dans l'algèbre des mots tassés. D'autre part, nous définissons un autre coproduit sur les forêts préordonnées donné par la contraction d'arêtes. Enfin, nous donnons une description combinatoire de morphismes définis sur des algèbres de Hopf de forêts et à valeurs dans les algèbres de Hopf de battages et de battages contractants.

Keywords. Algebraic combinatorics, planar rooted trees, Hopf algebra of ordered forests, quasi-shuffle algebra.

AMS Classification. 05E99, 16W30, 05C05.

Contents

Introduction

The Connes-Kreimer Hopf algebra of rooted forests $\mathbf{H}_{CK}$ is introduced and studied in [CK98, Moe01]. This commutative, noncocommutative Hopf algebra is used to study a problem of Renormalisation in Quantum Field Theory, as explained in [CK00, CK01]. The coproduct is given by admissible cuts. We denote by $\mathbf{H}^{\mathcal{D}}_{CK}$ the Hopf algebra of rooted trees, where the vertices are decorated by decorations belonging to the set $\mathcal{D}$. A noncommutative version, the Hopf algebra $\mathbf{H}_{NCK}$ of planar rooted forests, is introduced in [Foi02a, Foi02b, Hol03]. When the vertices are given a total order, we obtain the Hopf algebra of ordered forests $\mathbf{H}_o$ and, adding an increasing condition, we obtain the Hopf subalgebra of heap-ordered forests $\mathbf{H}_{ho}$ (see [FU12, GL90]).

On the other side, M.E. Hoffman studied in [Hof00] the Hopf algebra of shuffles $\mathrm{Sh}^{\mathcal{D}}$ and the Hopf algebra of quasi-shuffles $\mathbf{Csh}^{\mathcal{D}}$. In Mould Calculus theory, J. Ecalle and B. Vallet constructed a Hopf algebra morphism from $\mathbf{H}^{\mathcal{D}}_{CK}$ to $\mathrm{Sh}^{\mathcal{D}}$, or $\mathbf{Csh}^{\mathcal{D}}$, called the arborification morphism, or the contracting arborification morphism (see [EV04]). These morphisms are used for example in the problem of normal forms for vector fields to prove the convergence of series. We will describe in this paper all morphisms from $\mathbf{H}^{\mathcal{D}}_{CK}$ to $\mathrm{Sh}^{\mathcal{D}}$ or $\mathbf{Csh}^{\mathcal{D}}$.

First, to describe the morphisms from $\mathbf{H}^{\mathcal{D}}_{CK}$ to $\mathrm{Sh}^{\mathcal{D}}$, we introduce the notions of partition and contracted of a forest. This allows to obtain an already know coproduct called in this paper the contraction coproduct and defined by D. Calaque, K. Ebrahimi-Fard and D. Manchon on a quotient $\mathbf{C}_{CK}$ of $\mathbf{H}_{CK}$ (see [CEFM11, MS11]). We give a decorated version $\mathbf{C}^{\mathcal{D}}_{CK}$ of $\mathbf{C}_{CK}$. We define two operations $\curlyvee$ and $\triangleright$ on the vector space $\mathbf{T}^{\mathcal{D}}_{CK}$ spanned by the trees of $\mathbf{C}^{\mathcal{D}}_{CK}$. We prove that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ is a commutative prelie algebra, that is to say, $(A, \curlyvee)$ is a commutative algebra, $(A, \triangleright)$ is a prelie algebra and with the following relation: for all $x, y, z \in \mathbf{T}^{\mathcal{D}}_{CK}$,

$$x \triangleright (y \curlyvee z) = (x \triangleright y) \curlyvee z + (x \triangleright z) \curlyvee y.$$

We prove that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ is generated as commutative prelie algebra by the trees $\bullet_d$, $d \in \mathcal{D}$.

To describe the morphisms from $\mathbf{H}^{\mathcal{D}}_{CK}$ to $\mathbf{Csh}^{\mathcal{D}}$, we introduce the notion of preordered forests. There are rooted forests with a total preorder on their vertices (recall that a preorder is a binary reflexive and transitive relation). We prove that the algebra of preordered forests $\mathbf{H}_{po}$ is a Hopf algebra for the cut coproduct. With an increasing condition, we define the algebra of heap-preordered trees $\mathbf{H}_{hpo}$ and we prove that $\mathbf{H}_{hpo}$ is a Hopf subalgebra of $\mathbf{H}_{po}$. We construct a noncommutative version of the contraction Hopf algebra $\mathbf{C}_{CK}$. For this, we consider quotients of $\mathbf{H}_{NCK}$, $\mathbf{H}_{ho}$, $\mathbf{H}_o$, $\mathbf{H}_{hpo}$, $\mathbf{H}_{po}$, denoted respectively by $\mathbf{C}_{NCK}$, $\mathbf{C}_{ho}$, $\mathbf{C}_o$, $\mathbf{C}_{hpo}$, $\mathbf{C}_{po}$, and we define on these quotients a contraction coproduct. We prove that $\mathbf{C}_{ho}, \mathbf{C}_o, \mathbf{C}_{hpo}, \mathbf{C}_{po}$ are Hopf algebras and that $\mathbf{C}_{NCK}$ is a left comodule of the Hopf algebra $\mathbf{C}_{ho}$.

The Hopf algebra **FQSym** of free quasi-symmetric functions is introduced in [DHT02, MR95]. It is used by L. Foissy and J. Unterberger to give a noncommutative version of the arborification morphism in [FU12] (see also [Foi12]). They proved that there exists a Hopf algebra morphism from $\mathbf{H}_o$ to **FQSym** and that its restriction to $\mathbf{H}_{ho}$ is an isomorphism of Hopf algebras. We give in this paper an analogue of the contracting arborification morphism in the noncommutative case. For this, we substitute the ordered forests by the preordered forests and the quasi-symmetric functions by the packed words. Recall that the Hopf algebra $\mathbf{WQSym}^*$ of free packed words is a generalization of **FQSym** introduced by J.-C. Novelli and J.-Y. Thibon in [NT06]. Then we prove that there exists a Hopf algebra morphism from $\mathbf{H}_{po}$ to $\mathbf{WQSym}^*$. In addition, we prove that its restriction to $\mathbf{H}_{hpo}$ is an injection of Hopf algebras.

This text is organized as follows: the first section is devoted to reminders about the Hopf algebras, for the cut coproduct, of rooted forests, planar forests and ordered and heap-ordered forests. We give reminders on the Hopf algebras of words in Section 2. We define the Hopf algebra of permutations and packed words and we deduce the construction of $\mathrm{Sh}^{\mathcal{D}}$ and $\mathbf{Csh}^{\mathcal{D}}$. In Section 3, we define the algebras $\mathbf{H}_{po}$ and $\mathbf{H}_{hpo}$ of preordered and heap-preordered forests and we prove that these are Hopf algebras. The contraction coproduct, is introduced in Section 4. We describe a commutative case and we study an insertion operation. We give a noncommutative version using ordered and preordered forests. The last section deals with Hopf algebra morphisms from $\mathbf{H}_{CK}^{\mathcal{D}}$ or $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathrm{Sh}^{\mathcal{D}}$ or $\mathbf{Csh}^{\mathcal{D}}$. We give a combinatorial description of these morphisms in each case.

Acknowledgment. I would like to thank my advisor Loïc Foissy for stimulating discussions and his support during my research.

Notations.

1. We shall denote by $\mathbb{K}$ a commutative field of characteristic zero. Every vector space, algebra, coalgebra, etc, will be taken over $\mathbb{K}$. Given a set X, we denote by $\mathbb{K}\langle X\rangle$ the vector space spanned by X.

2. Let n be an integer. We denote by Σ_n the symmetric group of order n ($\Sigma_0 = \{1\}$) and Σ the disjoint union of Σ_n for all $n \geq 0$.

3. Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, nonzero, such that $\Delta(1) = 1 \otimes 1$. We then define the noncounitary coproduct:

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} \mathrm{Ker}(\varepsilon) & \to & \mathrm{Ker}(\varepsilon) \otimes \mathrm{Ker}(\varepsilon), \\ a & \mapsto & \Delta(a) - a \otimes 1 - 1 \otimes a. \end{array} \right.$$

1 Reminders on the Hopf algebras of forests

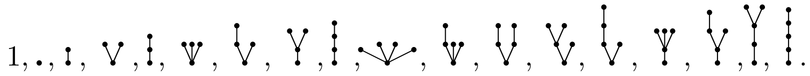
1.1 The Connes-Kreimer Hopf algebra of rooted trees

We briefly recall the construction of the Connes-Kreimer Hopf algebra of rooted trees [CK98]. A *rooted tree* is a finite graph, connected, without loops, with a distinguished vertex called the *root* [Sta02]. We denote by 1 the empty rooted tree. If T is a rooted tree, we denote by R_T the root of T. A *rooted forest* is a finite graph F such that any connected component of F is a rooted tree. The *length* of a forest F, denoted by $l(F)$, is the number of connected components of F. The set of vertices of the rooted forest F is denoted by $V(F)$. The *vertex degree* of a forest F, denoted by $|F|_v$, is the number of its vertices. The set of edges of the rooted forest F is denoted by $E(F)$. The *edge degree* of a forest F, denoted by $|F|_e$, is the number of its edges.

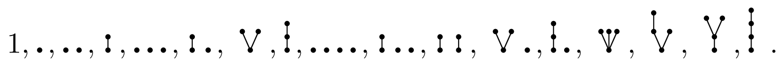
Remark. Let F be a rooted forest. Then $|F|_v = |F|_e + l(F)$.

Examples.

1. Rooted trees of vertex degree ≤ 5:

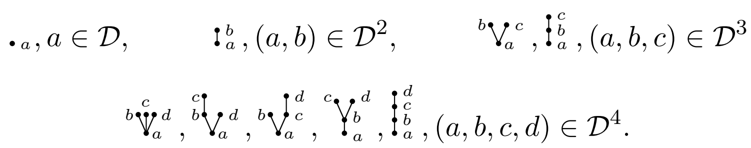


2. Rooted forests of vertex degree ≤ 4:



Let $\mathcal{D}$ be a nonempty set. A rooted forest with its vertices decorated by $\mathcal{D}$ is a pair (F, d) with F a rooted forest and $d : V(F) \to \mathcal{D}$ a map.

Examples. Rooted trees with their vertices decorated by $\mathcal{D}$ of vertex degree smaller than 4:



Let $\mathbb{F}_{\mathbf{H}_{CK}}$ be the set of rooted forests and $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$ the set of rooted forests with their vertices decorated by $\mathcal{D}$. We will denote by $\mathbf{H}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{CK}}$ and by $\mathbf{H}^{\mathcal{D}}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. The set of nonempty rooted trees will be denoted by $\mathbb{T}_{\mathbf{H}_{CK}}$ and the set of nonempty rooted trees with their vertices decorated by $\mathcal{D}$ will be denoted by $\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. $\mathbf{H}_{CK}$ and $\mathbf{H}^{\mathcal{D}}_{CK}$ are algebras: the product is given by the concatenation of rooted forests.

Let F be a nonempty rooted forest. A *subtree* T of F is a nonempty connected subgraph of F. A *subforest* $T_1 \dots T_k$ of F is a product of disjoint subtrees $T_1, \dots, T_k$ of F. We can give the same definition in the decorated case.

Examples. Consider the tree $T =$. Then:

- The subtrees of T are $\bullet$ (which appears 4 times), (which appears 3 times), , and (which appear once).
- The subforests of T are $\bullet\bullet$, (which appear 6 times), $\bullet$, $\bullet\bullet\bullet$ (which appear 4 times), , (which appear 3 times) and , , $\bullet\bullet\bullet\bullet$, , , (which appear once).

Let F be a rooted forest. The edges of F are oriented downwards (from the leaves to the roots). If $v, w \in V(F)$, we shall write $v \to w$ if there is an edge in F from v to w and $v \twoheadrightarrow w$ if there is an oriented path from v to w in F. By convention, $v \twoheadrightarrow v$ for any $v \in V(F)$.

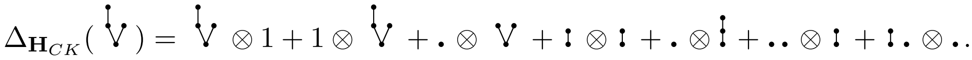
Let $\boldsymbol{v}$ be a subset of $V(F)$. We shall say that $\boldsymbol{v}$ is an admissible cut of F, and we shall write $\boldsymbol{v} \models V(F)$, if $\boldsymbol{v}$ is totally disconnected, that is to say, $v \not\twoheadrightarrow w$ for any pair (v, w) of two different elements of $\boldsymbol{v}$. If $\boldsymbol{v} \models V(F)$, we denote by $\mathrm{Lea}_{\boldsymbol{v}}(F)$ the rooted subforest of F obtained by keeping only the vertices above $\boldsymbol{v}$, that is to say, $\{w \in V(F),\ \exists v \in \boldsymbol{v},\ w \twoheadrightarrow v\}$, and the edges between these vertices. Note that $\boldsymbol{v} \subseteq \mathrm{Lea}_{\boldsymbol{v}}(F)$. We denote by $\mathrm{Roo}_{\boldsymbol{v}}(F)$ the rooted subforest obtained by keeping the other vertices and the edges between these vertices.

In particular, if $\boldsymbol{v} = \emptyset$, then $\mathrm{Lea}_{\boldsymbol{v}}(F) = 1$ and $\mathrm{Roo}_{\boldsymbol{v}}(F) = F$: this is the *empty cut* of F. If $\boldsymbol{v}$ contains all the roots of F, then it contains only the roots of F, $\mathrm{Lea}_{\boldsymbol{v}}(F) = F$ and $\mathrm{Roo}_{\boldsymbol{v}}(F) = 1$: this is the *total cut* of F. We shall write $\boldsymbol{v} \mid\models V(F)$ if $\boldsymbol{v}$ is a nontotal, nonempty admissible cut of F.

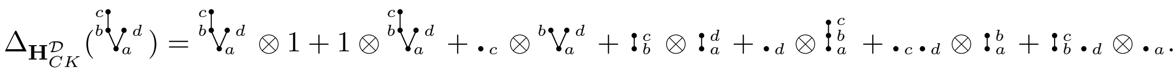
Connes and Kreimer proved in [CK98] that $\mathbf{H}_{CK}$ is a Hopf algebra. The coproduct is the cut coproduct defined for any rooted forest F by:

$$\Delta_{\mathbf{H}_{CK}}(F) = \sum_{\boldsymbol{v} \models V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \mid\models V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F).$$

For example:



In the same way, we can define a cut coproduct on $\mathbf{H}^{\mathcal{D}}_{CK}$. With this coproduct, $\mathbf{H}^{\mathcal{D}}_{CK}$ is also a Hopf algebra. For example:



$\mathbf{H}_{CK}$ is graded by the number of vertices. We give some values of the number $f_n^{\mathbf{H}_{CK}}$ of rooted forests of vertex degree n:

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842

This is the sequence A000081 in [Slo].

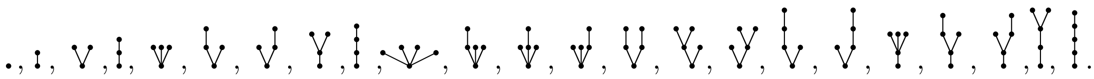
1.2 Hopf algebras of planar trees

We now recall the construction of the noncommutative generalization of the Connes-Kreimer Hopf algebra [Foi02a, Hol03].

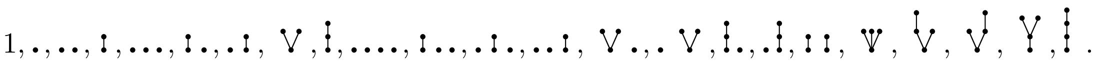
A *planar forest* is a rooted forest F such that the set of the roots of F is totally ordered and, for any vertex $v \in V(F)$, the set $\{w \in V(F) \mid w \to v\}$ is totally ordered. Planar forests are represented such that the total order on the set of roots and the sets $\{w \in V(F) \mid w \to v\}$ for any $v \in V(F)$ is given from left to right. We denote by $\mathbb{T}_{\mathbf{H}_{NCK}}$ the set of nonempty planar trees and $\mathbb{F}_{\mathbf{H}_{NCK}}$ the set of planar forests.

Examples.

1. Planar rooted trees of vertex degree ≤ 5:



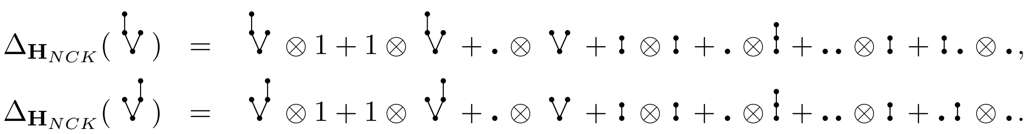
2. Planar rooted forests of vertex degree ≤ 4:



If $\boldsymbol{v} \models V(F)$, then $\mathrm{Lea}_{\boldsymbol{v}}(F)$ and $\mathrm{Roo}_{\boldsymbol{v}}(F)$ are naturally planar forests. It is proved in [Foi02a] that the space $\mathbf{H}_{NCK}$ generated by planar forests is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any rooted forest F by:

$$\Delta_{\mathbf{H}_{NCK}}(F) = \sum_{\boldsymbol{v} \models V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \Vdash V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F).$$

For example:



As in the nonplanar case, there is a decorated version $\mathbf{H}^{\mathcal{D}}_{NCK}$ of $\mathbf{H}_{NCK}$. Moreover, $\mathbf{H}_{NCK}$ is a Hopf algebra graded by the number of vertices. The number $f_n^{\mathbf{H}_{NCK}}$ of planar forests of vertex degree n (equal to the number of planar trees of vertex degree $n+1$) is the nth Catalan number $\frac{(2n)!}{n!(n+1)!}$, see sequence A000108 of [Slo]. We have:

$$T_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2}, \qquad F_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1}$$

This gives:

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{NCK}}$	1	1	2	5	14	42	132	429	1430	4862	16796

1.3 Ordered and heap-ordered forests

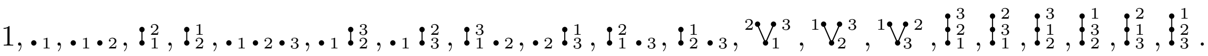
Definition 1 *An ordered forest F is a rooted forest F with a total order on $V(F)$. The set of ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_o}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_o}$ is denoted by $\mathbf{H}_o$.*

Remarks and notations. If F is an ordered forest, then there exits a unique increasing bijection $\sigma : V(F) \to \{1, \ldots, |F|_v\}$ for the total order on $V(F)$.

Conversely, if F is a rooted forest and if $\sigma : V(F) \to \{1, \ldots, |F|_v\}$ is a bijection, then σ defines a total order on $V(F)$ and F is an ordered forest.

Depending on the case, we shall denote an ordered forest by F or (F, σ).

Examples. Ordered forests of vertex degree ≤ 3:



Let (F, σ^F) and (G, σ^G) be two ordered forests. Then the rooted forest FG is also an ordered forest (FG, σ^{FG}) where

$$\sigma^{FG} = \sigma^F \otimes \sigma^G : \begin{cases} V(F) \bigcup V(G) & \to & \{1, \ldots, |F|_v + |G|_v\} \\ a \in V(F) & \mapsto & \sigma^F(a) \\ a \in V(G) & \mapsto & \sigma^G(a) + |F|_v . \end{cases} \tag{2}$$

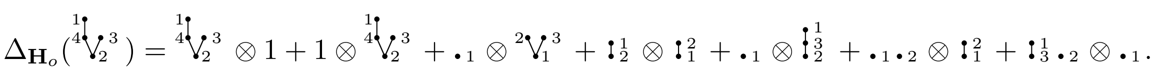
In other terms, we keep the initial order on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. This defines a noncommutative product on the set of ordered forests. For example, the product of $\bullet_1$ and $\vert^{2}_{1}$ gives $\bullet_1 \vert^{3}_{2} = \vert^{3}_{2} \bullet_1$, whereas the product of $\vert^{2}_{1}$ and $\bullet_1$ gives $\vert^{2}_{1} \bullet_3 = \bullet_3 \vert^{2}_{1}$. This product is linearly extended to $\mathbf{H}_o$, which in this way becomes an algebra.

$\mathbf{H}_o$ is graded by the number of vertices and there are $(n+1)^{n-1}$ ordered forests in vertex degree n, see sequence A000272 of [Slo].

If F is an ordered forest, then any subforest G of F is also ordered: the total order on $V(G)$ is the restriction of the total order of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_o} : \mathbf{H}_o \to \mathbf{H}_o \otimes \mathbf{H}_o$ on $\mathbf{H}_o$ in the following way: for all $F \in \mathbb{F}_{\mathbf{H}_o}$,

$$\Delta_{\mathbf{H}_o}(F) = \sum_{\boldsymbol{v} \models V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F).$$

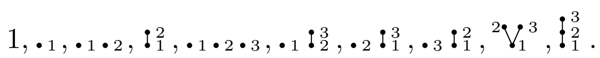
For example,



With this coproduct, $\mathbf{H}_o$ is a Hopf algebra.

Definition 2 *[GL90] A heap-ordered forest is an ordered forest F such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$, then a is greater than b for the total order on $V(F)$. The set of heap-ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{ho}}$.*

Examples. Heap-ordered forests of vertex degree ≤ 3:



Definition 3 *A linear order on a nonempty rooted forest F is a bijective map $\sigma : V(F) \to \{1, \ldots, |F|_v\}$ such that if $a, b \in V(F)$ and $a \twoheadrightarrow b$, then $\sigma(a) \geq \sigma(b)$. We denote by $\mathcal{O}(F)$ the set of linear orders on the nonempty rooted forest F.*

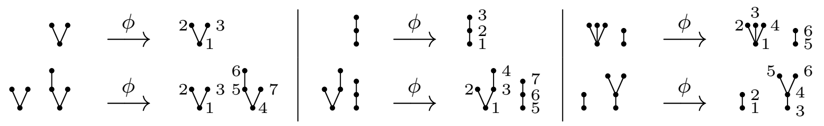
Remarks. If (F, σ) is a heap-ordered forest, then the increasing bijection $\sigma : V(F) \to \{1, \ldots, |F|_v\}$ is a linear order on F. Conversely, if F is a rooted forest and $\sigma \in \mathcal{O}(F)$, then σ defines a total order on $V(F)$ such that (F, σ) is a heap-ordered forest.

If F and G are two heap-ordered forests, then FG is an ordered forest with (2) and also a heap-ordered forest. Moreover, any subforest G of a heap-ordered forest F is also a heap-ordered forest by restriction on $V(G)$ of the total order of $V(F)$. So the subspace $\mathbf{H}_{ho}$ of $\mathbf{H}_o$ generated by the heap-ordered forests is a graded Hopf subalgebra of $\mathbf{H}_o$.

The number of heap-ordered forests of vertex degree n is $n!$, see sequence A000142 of [Slo].

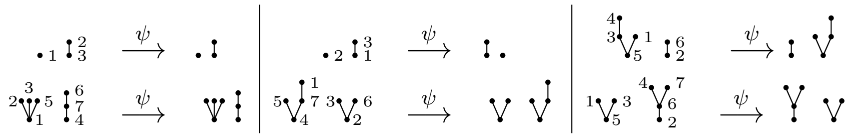
Remarks.

1. A planar forest can be considered as an ordered forest by ordering its vertices in the "north-west" direction, that is to say, from bottom to top and from left to right (this is the order defined in [Foi02a] or given by the Depth First Search algorithm). This defines an algebra morphism $\phi : \mathbf{H}_{NCK} \to \mathbf{H}_o$. For example:



(3)

2. Conversely, an ordered forest is also planar, by restriction of the total order to the subsets of vertices formed by the roots or $\{w \in V(\mathbb{F}) \mid w \to v\}$. This defines an algebra morphism $\psi : \mathbf{H}_o \to \mathbf{H}_{NCK}$. For example:



(4)

Note that $\psi \circ \phi = Id_{\mathbf{H}_{NCK}}$ therefore ψ is surjective and ϕ is injective. ψ and ϕ are not bijective (by considering the dimensions).

Moreover ϕ is a Hopf algebra morphism and its image is included in $\mathbf{H}_{ho}$. ψ is not a Hopf algebra morphism: in the expression of $(\psi \otimes \psi) \circ \Delta_{\mathbf{H}_o}(\cdot_3 \ldots)$ we have the tensor $\ldots \otimes \cdot$ and in the expression of $\Delta_{\mathbf{H}_{NCK}} \circ \psi(\cdot_3 \ldots)$ we have the different tensor $\ldots \otimes \cdot\cdot$. But the restriction of ψ of $\mathbf{H}_{ho}$ is a Hopf algebra morphism.

In the following, we consider $\mathbf{H}_{NCK}$ as a Hopf subalgebra of $\mathbf{H}_{ho}$ and $\mathbf{H}_o$.

2 Reminders on the Hopf algebras of words

2.1 Hopf algebra of permutations and shuffles

Notations.

1. Let k, l be integers. A (k,l)-*shuffle* is a permutation ζ of $\{1, \ldots, k+l\}$, such that $\zeta(1) < \ldots < \zeta(k)$ and $\zeta(k+1) < \ldots < \zeta(k+l)$. The set of (k,l)-shuffles will be denoted by $\mathrm{Sh}(k,l)$.

2. We represent a permutation $\sigma \in \Sigma_n$ by the word $(\sigma(1) \ldots \sigma(n))$. For example, $\mathrm{Sh}(2,1) = \{(123),\ (132),\ (231)\}$.

Remark. For any integer k, l, any permutation $\sigma \in \Sigma_{k+l}$ can be uniquely written as $\epsilon \circ (\sigma_1 \otimes \sigma_2)$, where $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in \mathrm{Sh}(k,l)$. Similarly, considering the inverses, any permutation $\tau \in \Sigma_{k+l}$ can be uniquely written as $(\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, where $\tau_1 \in \Sigma_k$, $\tau_2 \in \Sigma_l$, and $\zeta \in \mathrm{Sh}(k,l)$. Note that, whereas ϵ renames the numbers of each list $(\sigma(1), \ldots, \sigma(k)), (\sigma(k+1), \ldots, \sigma(k+l))$ without changing their orderings, ζ^{-1} shuffles the lists $(\tau(1), \ldots, \tau(k)), (\tau(k+1), \ldots, \tau(k+l))$. For instance, if $k = 4$, $l = 3$ and $\sigma = (5172436)$ then

- $\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)$, with $\epsilon = (1257346) \in \mathrm{Sh}(4,3)$, $\sigma_1 = (3142) \in \Sigma_4$ and $\sigma_2 = (213) \in \Sigma_3$,
- $\sigma = (\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, with $\tau_1 = (1243) \in \Sigma_4$, $\tau_2 = (132) \in \Sigma_3$ and $\zeta = (2456137) \in \mathrm{Sh}(4,3)$.

We here briefly recall the construction of the Hopf algebra $\mathbf{FQSym}$ of free quasi-symmetric functions, also called the Malvenuto-Reutenauer Hopf algebra [DHT02, MR95]. As a vector space, a basis of $\mathbf{FQSym}$ is given by the disjoint union of the symmetric groups Σ_n, for all $n \geq 0$. By convention, the unique element of Σ_0 is denoted by 1. The product of $\mathbf{FQSym}$ is given, for $\sigma \in \Sigma_k$, $\tau \in \Sigma_l$, by:

$$\sigma.\tau = \sum_{\zeta \in \mathrm{Sh}(k,l)} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example:

$$\begin{aligned} (123)(21) &= (12354) + (12534) + (15234) + (51234) + (12543) \\ &\quad + (15243) + (51243) + (15423) + (51423) + (54123). \end{aligned}$$

Let $\sigma \in \Sigma_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in \Sigma_k \times \Sigma_{n-k} \times \mathrm{Sh}(k, n-k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of $\mathbf{FQSym}$ is then defined by:

$$\Delta_{\mathbf{FQSym}}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2) \\ \epsilon \in \mathrm{Sh}(k,n-k), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then *standardizing* the two obtained words by the following process. If v is a words of length n whose the letters are distinct integers, then the standardizing of v, denoted by $\mathrm{Std}(v)$, is the word obtained by applying to the letters of v the unique increasing bijection to $\{1, \ldots, n\}$. For example:

$$\begin{aligned} \Delta_{\mathbf{FQSym}}((41325)) &= 1 \otimes (41325) + \mathrm{Std}(4) \otimes \mathrm{Std}(1325) + \mathrm{Std}(41) \otimes \mathrm{Std}(325) \\ &\quad + \mathrm{Std}(413) \otimes \mathrm{Std}(25) + \mathrm{Std}(4132) \otimes \mathrm{Std}(5) + (41325) \otimes 1 \\ &= 1 \otimes (41325) + (1) \otimes (1324) + (21) \otimes (213) \\ &\quad + (312) \otimes (12) + (4132) \otimes (1) + (41325) \otimes 1. \end{aligned}$$

Then **FQSym** is a Hopf algebra. It is graded, with $\mathbf{FQSym}(n) = Vect(\Sigma_n)$ for all $n \geq 0$.

It is also possible to give a decorated version of **FQSym**. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated permutation is a pair (σ, d), where $\sigma \in \Sigma_n$ and d is a map from $\{1, \ldots, n\}$ to $\mathcal{D}$. A $\mathcal{D}$-decorated permutation is represented by two superposed words $\binom{a_1 \ldots a_n}{v_1 \ldots v_n}$, where $(a_1 \ldots a_n)$ is the word representing σ and for all i, $v_i = d(a_i)$. The vector space $\mathbf{FQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated permutations is a Hopf algebra. For example, if $x, y, z, t \in \mathcal{D}$:

$$\begin{aligned}
\binom{213}{yxz} \cdot \binom{1}{t} &= \binom{2134}{yxzt} + \binom{2143}{yxtz} + \binom{2413}{ytxz} + \binom{4213}{tyxz}, \\
\Delta_{\mathbf{FQSym}^{\mathcal{D}}}\left(\binom{4321}{tzyx}\right) &= \binom{4321}{tzyx} \otimes 1 + \binom{321}{tzy} \otimes \binom{1}{x} + \binom{21}{tz} \otimes \binom{21}{yx} + \binom{1}{t} \otimes \binom{321}{zyx} + 1 \otimes \binom{4321}{tzyx}.
\end{aligned}$$

In other words, if (σ, d) and (τ, d') are decorated permutations of respective degrees k and l:

$$(\sigma, d).(\tau, d') = \sum_{\zeta \in \mathrm{Sh}(k,l)} ((\sigma \otimes \tau) \circ \zeta^{-1}, d \otimes d'),$$

where $d \otimes d'$ is defined by $(d \otimes d')(i) = d(i)$ if $1 \leq i \leq k$ and $(d \otimes d')(k+j) = d'(j)$ if $1 \leq j \leq l$. If (σ, d) is a decorated permutation of degree n:

$$\Delta_{\mathbf{FQSym}^{\mathcal{D}}}((\sigma, d)) = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2) \\ \epsilon \in \mathrm{Sh}(k,l), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_l}} (\sigma_1, d') \otimes (\sigma_2, d''),$$

where $d = (d' \otimes d'') \circ \epsilon^{-1}$.

In some sense, a $\mathcal{D}$-decorated permutation can be seen as a word with a total order on the set of its letters.

We can now define the shuffle Hopf algebra $\mathbf{Sh}^{\mathcal{D}}$ (see [Hof00, Reu93]). A $\mathcal{D}$-word is a finite sequence of elements taken in $\mathcal{D}$. It is graded by the degree of words, that is to say, the number of their letters. As a vector space, $\mathbf{Sh}^{\mathcal{D}}$ is generated by the set of $\mathcal{D}$-words.

The surjective linear map from $\mathbf{FQSym}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$, sending the decorated permutation $\binom{a_1 \ldots a_n}{v_1 \ldots v_n}$ to the $\mathcal{D}$-word $(v_1 \ldots v_n)$, defines a Hopf algebra structure on $\mathbf{Sh}^{\mathcal{D}}$:

- The product $\sqcup\!\sqcup$ of $\mathbf{Sh}^{\mathcal{D}}$ is given in the following way: if $(v_1 \ldots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1} \ldots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1 \ldots v_k) \sqcup\!\sqcup (v_{k+1} \ldots v_{k+l}) = \sum_{\zeta \in \mathrm{Sh}(k,l)} v_{\zeta^{-1}(1)} \ldots v_{\zeta^{-1}(k+l)}.$$

- The coproduct $\Delta_{\mathbf{Sh}^{\mathcal{D}}}$ of $\mathbf{Sh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $w = (v_1 \ldots v_n)$ by

$$\Delta_{\mathbf{Sh}^{\mathcal{D}}}(w) = \sum_{i=0}^{n} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n).$$

Examples.

1. If $(v_1 v_2 v_3)$ and $(v_4 v_5)$ are two $\mathcal{D}$-words,

$$\begin{aligned}
(v_1 v_2 v_3) \sqcup\!\sqcup (v_4 v_5) &= (v_1 v_2 v_3 v_4 v_5) + (v_1 v_2 v_4 v_3 v_5) + (v_1 v_2 v_4 v_5 v_3) + (v_1 v_4 v_2 v_3 v_5) \\
&\quad + (v_1 v_4 v_2 v_5 v_3) + (v_1 v_4 v_5 v_2 v_3) + (v_4 v_1 v_2 v_3 v_5) + (v_4 v_1 v_2 v_5 v_3) \\
&\quad + (v_4 v_1 v_5 v_2 v_3) + (v_4 v_5 v_1 v_2 v_3).
\end{aligned}$$

2. If $(v_1v_2v_3v_4)$ is a $\mathcal{D}$-word,

$$\begin{aligned}\Delta_{\mathrm{Sh}^{\mathcal{D}}}((v_1v_2v_3v_4)) &= (v_1v_2v_3v_4)\otimes 1 + (v_1v_2v_3)\otimes(v_4) + (v_1v_2)\otimes(v_3v_4)\\ &\quad +(v_1)\otimes(v_2v_3v_4) + 1\otimes(v_1v_2v_3v_4).\end{aligned}$$

There is a link between the algebras $\mathbf{H}_o$, $\mathbf{H}_{ho}$ and $\mathbf{FQSym}$ given by the following result (see [FU12]):

Proposition 4 *1. Let $n \geq 0$. For all $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_o}$, let $\mathbb{S}_F$ be the set of permutations $\theta \in \Sigma_n$ such that for all $a,b \in V(F)$, $(a \twoheadrightarrow b) \Rightarrow (\theta^{-1}(\sigma(a)) \leq \theta^{-1}(\sigma(b)))$. Let us define:*

$$\Theta : \left\{ \begin{array}{rcl} \mathbf{H}_o & \rightarrow & \mathbf{FQSym} \\ F \in \mathbb{F}_{\mathbf{H}_o} & \mapsto & \displaystyle\sum_{\theta \in \mathbb{S}_F} \theta. \end{array} \right.$$

Then $\Theta : \mathbf{H}_o \rightarrow \mathbf{FQSym}$ is a Hopf algebra morphism, homogeneous of degree 0.

2. The restriction of Θ to $\mathbf{H}_{ho}$ is an isomorphism of graded Hopf algebras.

2.2 Hopf algebra of packed words and quasi-shuffles

Recall the construction of the Hopf algebra $\mathbf{WQSym}^*$ of free packed words (see [NT06]).

Notations.

1. Let $n \geq 0$. We denote by Surj_n the set of maps $\sigma : \{1,\ldots,n\} \rightarrow \mathbb{N}^*$, such that $\sigma(\{1,\ldots,n\}) = \{1,\ldots,k\}$ for some $k \in \mathbb{N}$. In this case, k is the maximum of σ and is denoted by $\max(\sigma)$ and n is the length of σ. We represent the element σ of Surj_n by the packed word $(\sigma(1)\ldots\sigma(n))$.
2. Let k,l be two integers. A (k,l)-*surjective shuffle* is an element ϵ of Surj_{k+l} such that $\epsilon(1) < \ldots < \epsilon(k)$ and $\epsilon(k+1) < \ldots < \epsilon(k+l)$. The set of (k,l)-surjective shuffles will be denoted by $\mathrm{SjSh}(k,l)$. For example, $\mathrm{SjSh}(2,1) = \{(121),(122),(123),(132),(231)\}$.

Let v be a word such that the letters occuring in v are integers $a_1 < a_2 < \ldots < a_k$. The *packing* of v, denoted by $\mathrm{pack}(v)$, is the image of v by the map $a_i \mapsto i$. For example, $\mathrm{pack}((22539)) = (11324)$, $\mathrm{pack}((831535)) = (421323)$.

Remark. Let k,l be two integers and $\sigma \in \mathrm{Surj}_{k+l}$. We denote by $p_k = \max(\mathrm{pack}((\sigma(1)\ldots\sigma(k))))$ and $q_k = \max(\mathrm{pack}((\sigma(k+1)\ldots\sigma(k+l))))$. Then σ can be uniquely written as $\epsilon\circ(\sigma_1\otimes\sigma_2)$, where $\sigma_1 \in \mathrm{Surj}_k$, $\sigma_2 \in \mathrm{Surj}_l$, and $\epsilon \in \mathrm{SjSh}(p_k,q_k)$. For instance, if $k=4$, $l=3$ and $\sigma = (2311223)$ then $p_4 = 3$, $q_4 = 2$ and $\sigma = \epsilon\circ(\sigma_1\otimes\sigma_2)$ with $\epsilon = (12323) \in \mathrm{SjSh}(3,2)$, $\sigma_1 = (2311) \in \mathrm{Surj}_4$ and $\sigma_2 = (112) \in \mathrm{Surj}_3$.

As a vector space, a basis of $\mathbf{WQSym}^*$ is given by the disjoint union of the sets Surj_n, for all $n \geq 0$. By convention, the unique element of Surj_0 is denoted by 1. The product of $\mathbf{WQSym}^*$ is given, for $\sigma \in \mathrm{Surj}_k$ and $\tau \in \mathrm{Surj}_l$, by:

$$\sigma.\tau = \sum_{\zeta \in \mathrm{Sh}(k,l)} (\sigma\otimes\tau)\circ\zeta^{-1}.$$

In other words, as in the $\mathbf{FQSym}$ case, the product of σ and τ is given by shifting the letters of the word representing τ by k, and summing all the possible shuffings of this word and of the word representing σ. For example:

$$\begin{aligned}(112)(21) &= (11243)+(11423)+(14123)+(41123)+(11432)+(14132)\\ &\quad +(41132)+(14312)+(41312)+(43112)\end{aligned}$$

Let $\sigma \in \mathrm{Surj}_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in \mathrm{Surj}_k \times \mathrm{Surj}_{n-k} \times \mathrm{SjSh}(p_k, q_k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of $\mathbf{WQSym}^*$ is then given by:

$$\Delta_{\mathbf{WQSym}^*}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma=\epsilon\circ(\sigma_1\otimes\sigma_2) \\ \epsilon\in \mathrm{SjSh}(p_k,q_k), \sigma_1\in\mathrm{Surj}_k, \sigma_2\in\mathrm{Surj}_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words. For example:

$$\begin{aligned}\Delta_{\mathbf{WQSym}^*}((21132)) &= 1 \otimes (21132) + \mathrm{pack}((2)) \otimes \mathrm{pack}((1132)) + \mathrm{pack}((21)) \otimes \mathrm{pack}((132)) \\ &\quad + \mathrm{pack}((211)) \otimes \mathrm{pack}((32)) + \mathrm{pack}((2113)) \otimes \mathrm{pack}((2)) + (21132) \otimes 1 \\ &= 1 \otimes (21132) + (1) \otimes (1132) + (21) \otimes (132) + (211) \otimes (21) \\ &\quad + (2113) \otimes (1) + (21132) \otimes 1.\end{aligned}$$

Then $\mathbf{WQSym}^*$ is a graded Hopf algebra, with $\mathbf{WQSym}^*(n) = \mathrm{Surj}_n$ for all $n \geq 0$. We give some numerical values: if $f_n^{\mathbf{WQSym}^*}$ is the number of packed words of length n, then

n	0	1	2	3	4	5	6	7
$f_n^{\mathbf{WQSym}^*}$	1	1	3	13	75	541	4683	47293

These is the sequence A000670 in [Slo].

$\mathbf{WQSym}^*$ is the gradued dual of $\mathbf{WQSym}$, described as follows. A basis of $\mathbf{WQSym}$ is given by the disjoint union of the sets Surj_n. The product of $\mathbf{WQSym}$ is given, for $\sigma \in \mathrm{Surj}_k$, $\tau \in \mathrm{Surj}_l$ by:

$$\sigma.\tau = \sum_{\substack{\gamma=\gamma_1\ldots\gamma_{k+l} \\ \mathrm{pack}(\gamma_1\ldots\gamma_k)=\sigma,\ \mathrm{pack}(\gamma_{k+1}\ldots\gamma_{k+l})=\tau}} \gamma$$

In other terms, the product of σ and τ is given by shifting certain letters of the words representing σ and τ and then summing all concatenations of obtained words. For example:

$$\begin{aligned}(112)(21) &= (11221) + (11321) + (22321) + (33421) + (11231) + (22331) + (22431) \\ &\quad + (11232) + (11332) + (11432) + (22341) + (11342) + (11243)\end{aligned}$$

If $\sigma \in \mathrm{Surj}_n$, then the coproduct of $\mathbf{WQSym}$ is given by:

$$\Delta_{\mathbf{WQSym}}(\sigma) = \sum_{0\leq k\leq \max(\sigma)} \sigma_{|[1,k]} \otimes \mathrm{pack}(\sigma_{|[k+1,\max(\sigma)]}),$$

where $\sigma_{|\mathcal{A}}$ is the subword obtained by tacking in σ the letters from the subset $\mathcal{A}$ of $[1, \max(\sigma)]$. For example:

$$\begin{aligned}\Delta_{\mathbf{WQSym}}((21312245)) &= 1 \otimes (21312245) + (11) \otimes \mathrm{pack}((232245)) + (21122) \otimes \mathrm{pack}((345)) \\ &\quad + (213122) \otimes \mathrm{pack}((45)) + (2131224) \otimes \mathrm{pack}((5)) + (21312245) \otimes 1 \\ &= 1 \otimes (21312245) + (11) \otimes (121134) + (21122) \otimes (123) \\ &\quad + (213122) \otimes (12) + (2131224) \otimes (1) + (21312245) \otimes 1.\end{aligned}$$

Then $\mathbf{WQSym}$ is a Hopf algebra.

We give a decorated version of $\mathbf{WQSym}$. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated surjection is a pair (σ, d), where $\sigma \in \mathrm{Surj}_n$ and d is a map from $\{1, \ldots, n\}$ to $\mathcal{D}$. As in the $\mathbf{FQSym}^{\mathcal{D}}$ case, we represent a $\mathcal{D}$-decorated surjection by two superposed words $\binom{a_1\ldots a_n}{v_1\ldots v_n}$, where $(a_1\ldots a_n)$ is the packed word representing σ and for all i, $v_i = d(i)$. The vector space $\mathbf{WQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated surjections is a Hopf algebra. For example, if $x, y, z, t \in \mathcal{D}$:

$$\begin{array}{rcl} \binom{211}{yxz}\cdot\binom{1}{t} & = & \binom{2111}{yxzt}+\binom{2112}{yxzt}+\binom{2113}{yxzt}+\binom{3221}{yxzt}+\binom{3112}{yxzt}\,. \\ \Delta_{\mathbf{WQSym}^{\mathcal{D}}}\left(\binom{2113}{yxzt}\right) & = & \binom{2113}{yxzt}\otimes 1+\binom{11}{xz}\otimes\binom{12}{yt}+\binom{211}{yxz}\otimes\binom{1}{t}+1\otimes\binom{2113}{yxzt}\,. \end{array}$$

In other words, if (σ, d) and (τ, d') are decorated surjections of respective degrees k and l:

$$(\sigma,d).(\tau,d') = \sum_{\substack{\gamma=\gamma_1\ldots\gamma_{k+l}\\ \mathrm{pack}(\gamma_1\ldots\gamma_k)=\sigma,\ \mathrm{pack}(\gamma_{k+1}\ldots\gamma_{k+l})=\tau}} (\gamma, d\otimes d'),$$

where $d\otimes d'$ is defined by $(d\otimes d')(i) = d(i)$ if $1\le i\le k$ and $(d\otimes d')(k+j) = d'(j)$ if $1\le j\le l$. If (σ, d) is a decorated surjection of degree n:

$$\Delta_{\mathbf{WQSym}^{\mathcal{D}}}((\sigma,d)) = \sum_{0\le k\le \max(\sigma)} (\sigma_{|[1,k]}, d')\otimes(\sigma_{|[k+1,\max(\sigma)]}, d'').$$

where d' and d take the same values on $\sigma^{-1}(\{1,\ldots,k\})$ and d'' and d take the same values on $\sigma^{-1}(\{k+1,\ldots,\max(\sigma)\})$.

In some sense, a $\mathcal{D}$-decorated surjection can be seen as a packed word with a preorder on the set of its letters.

Suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b)\in\mathcal{D}^2 \to [ab]\in\mathcal{D}$. We define by induction $[\cdot,\cdot]^{(k)}$:

$$[\cdot,\cdot]^{(0)} = Id,\ [\cdot,\cdot]^{(1)} = [\cdot,\cdot] \text{ and } [\cdot,\cdot]^{(k)} = \left[\cdot,[\cdot,\cdot]^{(k-1)}\right].$$

We can now define the quasi-shuffle Hopf algebra $\mathbf{Csh}^{\mathcal{D}}$ (see [Hof00]). $\mathbf{Csh}^{\mathcal{D}}$ is, as a vector space, generated by the set of $\mathcal{D}$-words.

Let φ be the surjective linear map from $\mathbf{WQSym}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$ defined, for (σ,d) a decorated surjection of maximum k, by $\varphi((\sigma,d)) = (w_1\ldots w_k)$ where $w_j = [d(i_1)\ldots d(i_p)]^{(p)}$ with $\sigma^{-1}(j) = \{i_1,\ldots,i_p\}$. For instance,

$$\varphi\left(\binom{2114324}{yxztvwu}\right) = ([xz]\ [yw]\ v\ [tu])$$

Notations. Let k,l be integers. A (k,l)-quasi-shuffle of type r is a surjective map $\zeta : \{1,\ldots,k+l\} \twoheadrightarrow \{1,\ldots,k+l-r\}$ such that

$$\begin{cases} \zeta(1)<\ldots<\zeta(k), \\ \zeta(k+1)<\ldots<\zeta(k+l). \end{cases}$$

Remark that $\zeta^{-1}(j)$ contains 1 or 2 elements for all $1\le j\le k+l-r$. The set of (k,l)-quasi-shuffles of type r is denoted by $\mathrm{Csh}(p,q,r)$. Remark that $\mathrm{Csh}(k,l,0) = \mathrm{Sh}(k,l)$.

φ define a Hopf algebra structure on $\mathbf{Csh}^{\mathcal{D}}$:

- The product $\uplus$ of $\mathbf{Csh}^{\mathcal{D}}$ is given in the following way: if $(v_1\ldots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1}\ldots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1\ldots v_k)\uplus(v_{k+1}\ldots v_{k+l}) = \sum_{r\ge 0}\sum_{\zeta\in\mathrm{Csh}(k,l,r)} (w_1\ldots w_{k+l-r}),$$

  where $w_j = v_i$ if $\zeta^{-1}(j) = \{i\}$ and $w_j = [v_{i_1}v_{i_2}]$ if $\zeta^{-1}(j) = \{i_1,i_2\}$.

- The coproduct $\Delta_{\mathbf{Csh}^{\mathcal{D}}}$ of $\mathbf{Csh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $v = (v_1 \dots v_n)$ by

  $$\Delta_{\mathbf{Csh}^{\mathcal{D}}}(v) = \sum_{i=0}^{n} (v_1 \dots v_i) \otimes (v_{i+1} \dots v_n).$$

  This is the same coproduct as for $\mathbf{Sh}^{\mathcal{D}}$.

Example. If $(v_1 v_2)$ and $(v_3 v_4)$ are two $\mathcal{D}$-words,

$$\begin{array}{rcl}(v_1 v_2) \uplus (v_3 v_4) & = & (v_1 v_2 v_3 v_4) + (v_1 v_3 v_2 v_4) + (v_3 v_1 v_2 v_4) + (v_1 v_3 v_4 v_2) \\ & & +(v_3 v_1 v_4 v_2) + (v_3 v_4 v_1 v_2) + (v_1 \, [v_2 v_3] \, v_4) + ([v_1 v_3] \, v_2 v_4) \\ & & +(v_1 v_3 \, [v_2 v_4]) + (v_3 \, [v_1 v_4] \, v_2) + ([v_1 v_3] \, [v_2 v_4])\end{array}$$

3 Preordered forests

3.1 Preordered and heap-preordered forests

A preorder is a binary, reflexive and transitive relation. In particular, an antisymmetric preorder is an order. A preorder is total if two elements are always comparable. We introduce another version of ordered forests, the preordered forests.

Definition 5 *A preordered forest F is a rooted forest F with a total preorder on $V(F)$. The set of preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{po}}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{po}}$ is denoted by $\mathbf{H}_{po}$.*

Remarks and notations.

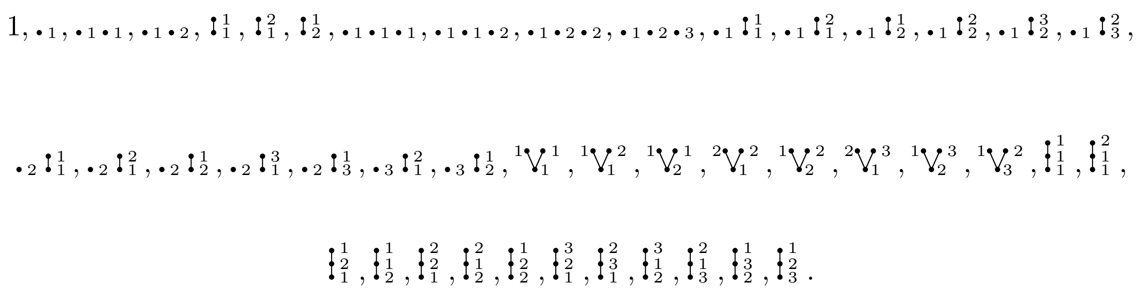
1. Let F be a preordered forest. We denote by $\leq$ the total preorder on $V(F)$. Remark that the antisymmetric relation "$x \leq y$ and $y \leq x$" is an equivalence relation denoted by $\mathcal{R}$ and the quotient set $V(F)/\mathcal{R}$ is totally ordered. We denote by q the cardinality of this quotient set. Let $\overline{\sigma}$ be the unique increasing map from $V(F)/\mathcal{R}$ to $\{1, \dots, q\}$. There exists a unique surjection $\sigma : V(F) \to \{1, \dots, q\}$, compatible with the equivalence $\mathcal{R}$, such that the induced map on $V(F)/\mathcal{R}$ is $\overline{\sigma}$. In the sequel, we shall write $q = \max(F)$ (and we have always $q \leq |F|_v$).

   Conversely, if F is a rooted forest and if $\sigma : V(F) \to \{1, \dots, q\}$ is a surjection, $q \leq |F|_v$, then σ define a total preorder on $V(F)$ and F is a preordered forest.

   As in the ordered case, we shall denote a preordered forest by F or (F, σ).

2. An ordered forest is also a preordered forest. Conversely, a preordered forest (F, σ) is an ordered forest if $|F|_v = \max(F)$.

Examples. Preordered forests of vertex degree ≤ 3:



Let (F,σ^F) and (G,σ^G) be two preordered forests with $\sigma^F : V(F) \to \{1,\ldots,q\}$, $\sigma^G : V(G) \to \{1,\ldots,r\}$, $q = \max(F)$ and $r = \max(G)$. Then FG is also a preordered forest (FG,σ^{FG}) where

$$\sigma^{FG} = \sigma^F \otimes \sigma^G : \left\{ \begin{array}{rcl} V(F)\bigcup V(G) & \to & \{1,\ldots,q+r\} \\ a \in V(F) & \mapsto & \sigma^F(a) \\ a \in V(G) & \mapsto & \sigma^G(a)+q. \end{array} \right. \tag{5}$$

In other terms, we keep the initial preorder on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. In this way, we define a noncommutative product on the set of preordered forests. For example, the product of $\mathbf{I}^{3}_{1}\bullet_2$ and ${}^1\mathbf{V}^2_1$ gives $\mathbf{I}^{3}_{1}\bullet_2\,{}^4\mathbf{V}^5_4$, whereas the product of ${}^1\mathbf{V}^2_1$ and $\mathbf{I}^{3}_{1}\bullet_2$ gives ${}^1\mathbf{V}^2_1\,\mathbf{I}^{5}_{3}\bullet_4$. Remark that, if F and G are two preordered forests, $\max(FG) = \max(F)+\max(G)$. This product is linearly extended to $\mathbf{H}_{po}$, which in this way becomes an algebra, gradued by the number of vertices.

Remark. The formula (5) on the preordered forests extends the formula (2) on the ordered forests.

We give some numerical values: if $f_n^{\mathbf{H}_{po}}$ is the number of preordered forests of vertex degree n,

n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{po}}$	1	1	5	38	424	6284

If F is a preordered forest, then any subforest G of F is also preordered: the total preorder on $V(G)$ is the restriction of the total preorder of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_{po}} : \mathbf{H}_{po} \to \mathbf{H}_{po} \otimes \mathbf{H}_{po}$ on $\mathbf{H}_{po}$ in the following way: for all $F \in \mathbb{F}_{\mathbf{H}_{po}}$,

$$\Delta_{\mathbf{H}_{po}}(F) = \sum_{\boldsymbol{v} \models V(F)} \mathrm{Lea}_{\boldsymbol{v}}(F) \otimes \mathrm{Roo}_{\boldsymbol{v}}(F).$$

For example,

$$\Delta_{\mathbf{H}_{po}}\left({}^{1}_{2}\mathbf{V}^{3}_{1}\right) = {}^{1}_{2}\mathbf{V}^{3}_{1} \otimes 1 + 1 \otimes {}^{1}_{2}\mathbf{V}^{3}_{1} + \bullet_1 \otimes {}^2\mathbf{V}^3_1 + \mathbf{I}^{1}_{2} \otimes \mathbf{I}^{2}_{1} + \bullet_1 \otimes \mathbf{I}^{1}_{2\,1} + \bullet_1\bullet_2 \otimes \mathbf{I}^{2}_{1} + \mathbf{I}^{1}_{2}\bullet_3 \otimes \bullet_1.$$

With this coproduct, $\mathbf{H}_{po}$ is a Hopf algebra. Remark that $\mathbf{H}_o$ is a Hopf subalgebra of $\mathbf{H}_{po}$.

Definition 6 *A heap-preordered forest is a preordered forest F such that if $a,b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$, then a is strictly greater than b for the total preorder on $V(F)$. The set of heap-preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{hpo}}$*

Examples. Heap-preordered forests of vertex degree ≤ 3:

$$1, \bullet_1, \bullet_1\bullet_1, \bullet_1\bullet_2, \mathbf{I}^{2}_{1}, \bullet_1\bullet_1\bullet_1, \bullet_1\bullet_1\bullet_2, \bullet_1\bullet_2\bullet_2, \bullet_1\bullet_2\bullet_3, \bullet_1\mathbf{I}^{2}_{1}, \bullet_1\mathbf{I}^{3}_{2}, \bullet_2\mathbf{I}^{2}_{1}, \bullet_2\mathbf{I}^{3}_{1}, \bullet_3\mathbf{I}^{2}_{1}, {}^2\mathbf{V}^2_1, {}^2\mathbf{V}^3_1, \mathbf{I}^{3}_{2\,1}.$$

Definition 7 *Let F be a nonempty rooted forest and q an integer $\leq |F|_v$. A linear preorder is a surjection $\sigma : V(F) \to \{1,\ldots,q\}$ such that if $a,b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$ then $\sigma(a) > \sigma(b)$. We denote by $\mathcal{O}_p(F)$ the set of linear preorders on the nonempty rooted forest F.*

Remarks. If (F,σ) is a heap-preordered forest, the surjection $\sigma : V(F) \to \{1,\ldots,\max(F)\}$ is a linear preorder on F. Conversely, if F is a rooted forest and $\sigma \in \mathcal{O}_p(F)$, then σ define a total preorder on $V(F)$ such that (F,σ) is a heap-preordered forest.

If F and G are two heap-preordered forests, then FG is also heap-preordered. Moreover, any subforest G of a heap-preordered forest F is also a heap-preordered forest by restriction on $V(G)$ of the total preorder of $V(F)$. So the subspace $\mathbf{H}_{hpo}$ of $\mathbf{H}_{po}$ generated by the heap-preordered forests is a graded Hopf subalgebra of $\mathbf{H}_{po}$.

We give some numerical values: if $f_n^{\mathbf{H}_{hpo}}$ is the number of preordered forests of vertex degree n,

n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{hpo}}$	1	1	3	12	64	428

We have the following diagram

$$\begin{tikzcd} \mathbf{H}_{NCK} \arrow[r, hook] & \mathbf{H}_{ho} \arrow[r, hook] \arrow[d, hook] & \mathbf{H}_{o} \arrow[d, hook] \\ & \mathbf{H}_{hpo} \arrow[r, hook] & \mathbf{H}_{po} \end{tikzcd} \tag{6}$$

where the arrows $\hookrightarrow$ are injective morphisms of Hopf algebras (for the cut coproduct).

3.2 A morphism from $\mathbf{H}_{po}$ to $\mathbf{WQSym}^*$

In this section, we give a result similar to proposition 4 in the preordered case.

Definition 8 *Let (F,σ) be a nonempty preordered forest of vertex degree n and $\tau \in \mathrm{Surj}_n$. Then we denote by $\mathbb{S}^{\tau}_{(F,\sigma)}$ the set of bijective maps $\varphi : V(F) \to \{1,\ldots,n\}$ such that:*

1. *if $v \in V(F)$, then $\sigma(v) = \tau(\varphi(v))$,*
2. *if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.*

Remark.

1. If $\max(F) \neq \max(\tau)$, then $\mathbb{S}^{\tau}_{(F,\sigma)} = \emptyset$.
2. Let $F, G \in \mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v = k$, $|G|_v = l$. If $\varphi_1 : V(F) \to \{1,\ldots,k\}$ and $\varphi_2 : V(G) \to \{1,\ldots,l\}$ are two bijective maps and $\zeta \in \mathrm{Sh}(k,l)$, then $\zeta \circ (\varphi_1 \otimes \varphi_2) : V(FG) \to \{1,\ldots,k+l\}$, where $\varphi_1 \otimes \varphi_2$ is defined in formula (2), is also a bijective map. Similary, considering a bijective map $\varphi : V(FG) \to \{1,\ldots,k+l\}$ and $\zeta \in \mathrm{Sh}(k,l)$. Then φ can be uniquely written as $\zeta \circ (\varphi_1 \otimes \varphi_2)$, where $\varphi_1 : V(F) \to \{1,\ldots,k\}$ and $\varphi_2 : V(G) \to \{1,\ldots,l\}$ are two bijective maps.

Theorem 9 *Let us define:*

$$\Phi : \left\{ \begin{array}{rcl} \mathbf{H}_{po} & \to & \mathbf{WQSym}^* \\ (F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}} & \mapsto & \displaystyle\sum_{\tau \in \mathrm{Surj}_{|F|_v}} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau. \end{array} \right. \tag{7}$$

Then $\Phi : \mathbf{H}_{po} \to \mathbf{WQSym}^*$ is a Hopf algebra morphism, homogeneous of degree 0.

Examples.

- In vertex degree 1: $\Phi(\bullet_1) = (1)$.
- In vertex degree 2:

$$\Phi(\bullet_1\bullet_1) = 2(11), \qquad \Phi(\bullet_1\bullet_2) = (12) + (21), \qquad \Phi(\ell^{a}_{b}) = (ab).$$

- In vertex degree 3:

$$\begin{array}{rclrcl}
\Phi(\bullet_1\bullet_1\bullet_1) & = & 6(111) & \Phi(\ell^{2}_{1}\bullet_2) & = & (212) + 2(221) \\
\Phi({}^{2}\mathrm{V}^{1}_{2}) & = & (122) + (212) & \Phi(\bullet_2\,\ell^{1}_{3}) & = & (213) + (123) + (132) \\
\Phi(\ell^{2}_{\substack{3\\1}}) & = & (231) & \Phi(\bullet_1\,\ell^{2}_{3}) & = & (123) + (213) + (231) \\
\Phi({}^{2}\mathrm{V}^{2}_{1}) & = & 2(221) & \Phi(\bullet_1\bullet_1\bullet_2) & = & 2\left[(112) + (121) + (211)\right]
\end{array}$$

Proof. Obviously, Φ is homogeneous of degree 0. Let $(F, \sigma^F), (G, \sigma^G) \in \mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v = k$, $|G|_v = l$ and $\tau \in \mathrm{Surj}_{k+l}$. τ can be uniquely written as $\tau = (\tau_1 \otimes \tau_2) \circ \zeta^{-1}$ with $\tau_1 \in \mathrm{Surj}_k$, $\tau_2 \in \mathrm{Surj}_l$ and $\zeta \in \mathrm{Sh}(k, l)$.

Let $\varphi \in \mathbb{S}^{\tau}_{(FG, \sigma^{FG})}$. Then φ can be uniquely written as $\zeta \circ (\varphi_1 \otimes \varphi_2)$ with $\varphi_1 : V(F) \to \{1, \ldots, k\}$ and $\varphi_2 : V(G) \to \{1, \ldots, l\}$ two bijective maps.

1. (a) If $v \in V(F)$, then

$$\sigma^F(v) = \sigma^{FG}(v) = \tau(\varphi(v)) = (\tau_1 \otimes \tau_2) \circ \zeta^{-1} \circ \zeta \circ (\varphi_1 \otimes \varphi_2)(v) = \tau_1(\varphi_1(v)).$$

Note that with this equality, we also have that $\max(F) = \max(\tau_1)$.

(b) If $v \in V(G)$, then

$$\begin{array}{rcl}
\sigma^G(v) + \max(F) & = & \sigma^{FG}(v) = \tau(\varphi(v)) = (\tau_1 \otimes \tau_2) \circ \zeta^{-1} \circ \zeta \circ (\varphi_1 \otimes \varphi_2)(v) \\
& = & \tau_2(\varphi_2(v)) + \max(\tau_1).
\end{array}$$

As $\max(F) = \max(\tau_1)$, $\sigma^G(v) = \tau_2(\varphi_2(v))$.

2. (a) If $v' \twoheadrightarrow v$ in F, then $v' \twoheadrightarrow v$ in FG, so:

$$\begin{array}{rcl}
\varphi(v) & \geq & \varphi(v') \\
\zeta \circ (\varphi_1 \otimes \varphi_2)(v) & \geq & \zeta \circ (\varphi_1 \otimes \varphi_2)(v') \\
\zeta\left(\varphi_1(v)\right) & \geq & \zeta\left(\varphi_1(v')\right) \\
\varphi_1(v) & \geq & \varphi_1(v'),
\end{array}$$

as ζ is increasing on $\{1, \ldots, k\}$.

(b) If $v' \twoheadrightarrow v$ in G, then $v' \twoheadrightarrow v$ in FG, so:

$$\begin{array}{rcl}
\varphi(v) & \geq & \varphi(v') \\
\zeta \circ (\varphi_1 \otimes \varphi_2)(v) & \geq & \zeta \circ (\varphi_1 \otimes \varphi_2)(v') \\
\zeta\left(k + \varphi_2(v)\right) & \geq & \zeta\left(k + \varphi_2(v')\right) \\
\varphi_2(v) & \geq & \varphi_2(v'),
\end{array}$$

as ζ is increasing on $\{k+1, \ldots, k+l\}$.

So $\varphi_1 \in \mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2 \in \mathbb{S}^{\tau_2}_{(G,\sigma^G)}$.

Conversely, if $\varphi = \zeta \circ (\varphi_1 \otimes \varphi_2)$, with $\varphi_1 \in \mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2 \in \mathbb{S}^{\tau_2}_{(G,\sigma^G)}$, the same computations shows that $\varphi \in \mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})}$.

So
$$\mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})}) = \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)}) \times \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})$$
and
$$\begin{aligned}
\Phi((FG,\sigma^{FG})) &= \sum_{\tau\in\mathrm{Surj}_{k+l}} \mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})})\ \tau\\
&= \sum_{\zeta\in\mathrm{Sh}(k,l)} \sum_{\tau_1\in\mathrm{Surj}_k} \sum_{\tau_2\in\mathrm{Surj}_l} \mathrm{card}(\mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})})\ (\tau_1\otimes\tau_2)\circ\zeta^{-1}\\
&= \sum_{\zeta\in\mathrm{Sh}(k,l)} \sum_{\tau_1\in\mathrm{Surj}_k} \sum_{\tau_2\in\mathrm{Surj}_l} \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)}) \times \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\ (\tau_1\otimes\tau_2)\circ\zeta^{-1}\\
&= \left(\sum_{\tau_1\in\mathrm{Surj}_k} \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)})\ \tau_1\right)\left(\sum_{\tau_2\in\mathrm{Surj}_l} \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\ \tau_2\right)\\
&= \Phi((F,\sigma^F))\Phi((G,\sigma^G)).
\end{aligned}$$

So Φ is an algebra morphism.

Let $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest such that $|F|_v = n$ and let $\boldsymbol{v}$ be an admissible cut of F. We obtain two preordered forests $(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1)$ and $(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2)$. We denote by $k = |\mathrm{Lea}_{\boldsymbol{v}}(F)|_v$ and $l = |\mathrm{Roo}_{\boldsymbol{v}}(F)|_v$.

Let $\tau_1 \in \mathrm{Surj}_k$, $\tau_2 \in \mathrm{Surj}_l$ and $\varphi_1 \in \mathbb{S}^{\tau_1}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_2 \in \mathbb{S}^{\tau_2}_{(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2)}$. We denote by $\varphi = \varphi_1\otimes\varphi_2$ and we define τ by $\tau = \sigma\circ\varphi^{-1}$. $\tau \in \mathrm{Surj}_n$ and $\max(\tau) = \max(F)$. Let us show that $\varphi \in \mathbb{S}^{\tau}_{(F,\sigma)}$.

1. By definition, $\tau = \sigma\circ\varphi^{-1}$. So $\sigma(v) = \tau(\varphi(v))$ for all $v \in V(F)$.
2. If $v' \twoheadrightarrow v$ in F, then three cases are possible:
   (a) v and v' belong to $V(\mathrm{Lea}_{\boldsymbol{v}}(F))$. As $\varphi_1 \in \mathbb{S}^{\tau_1}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_1(v) \geq \varphi_1(v')$. Then $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_1(v) \geq \varphi_1(v') = (\varphi_1\otimes\varphi_2)(v') = \varphi(v')$.
   (b) v and v' belong to $V(\mathrm{Roo}_{\boldsymbol{v}}(F))$. As $\varphi_2 \in \mathbb{S}^{\tau_2}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_2)}$, $\varphi_2(v) \geq \varphi_2(v')$. Then $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_2(v) + k \geq \varphi_2(v') + k = (\varphi_1\otimes\varphi_2)(v') = \varphi(v')$.
   (c) v' belong to $V(\mathrm{Lea}_{\boldsymbol{v}}(F))$ and v belong to $V(\mathrm{Roo}_{\boldsymbol{v}}(F))$. Then $\varphi(v') = (\varphi_1\otimes\varphi_2)(v') = \varphi_1(v') \in \{1,\dots,k\}$ and $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_2(v) + k \in \{k+1,\dots,k+l\}$. So $\varphi(v) > \varphi(v')$.

   In any case, $\varphi(v) \geq \varphi(v')$.

Conversely, let $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest of vertex degree n, $\tau \in \mathrm{Surj}_n$ and $\varphi \in \mathbb{S}^{\tau}_{(F,\sigma)}$. Let $k \in \{0,\dots,n\}$ be an integer.

We denote by $\tau_1^{(k)}$ and $\tau_2^{(k)}$ the words obtained by cutting the word representing τ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words.

Moreover, we define $\boldsymbol{v}$ a subset of $\varphi^{-1}(\{1,\dots,k\})$ such that $v \not\twoheadrightarrow w$ for any pair (v,w) of two different elements of $\boldsymbol{v}$. Then $\boldsymbol{v} \models V(F)$ and we consider the two preordered forests $(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1^{(k)})$ and $(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2^{(k)})$. Remark that, with the second point of definition 8, $V(\mathrm{Lea}_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{1,\dots,k\})$ and $V(\mathrm{Roo}_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{k+1,\dots,n\})$.

We denote by $\varphi_1^{(k)} : v \in V(\mathrm{Lea}_{\boldsymbol{v}}(F)) \to \varphi(v) \in \{1,\dots,k\}$ and $\varphi_2^{(k)} : v \in V(\mathrm{Roo}_{\boldsymbol{v}}(F)) \to \varphi(v) - k \in \{1,\dots,n-k\}$. Thus $\varphi = \varphi_1^{(k)} \otimes \varphi_2^{(k)}$.

Let us prove that $\varphi_1^{(k)} \in \mathbb{S}^{\tau_1^{(k)}}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1^{(k)})}$ and $\varphi_2^{(k)} \in \mathbb{S}^{\tau_2^{(k)}}_{(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2^{(k)})}$.

1. (a) If $v \in V(\mathrm{Lea}_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_1^{(k)}(v) \in \{1,\dots,k\}$ and then

$$\sigma_1^{(k)}(v) = \mathrm{pack} \circ \sigma(v) = \mathrm{pack} \circ \tau \circ \varphi(v) = \tau_1^{(k)} \circ \varphi_1^{(k)}(v).$$

   (b) If $v \in V(\mathrm{Roo}_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_2^{(k)}(v) + k \in \{k+1,\dots,n\}$ and then

$$\sigma_2^{(k)}(v) = \mathrm{pack} \circ \sigma(v) = \mathrm{pack} \circ \tau \circ \varphi(v) = \tau_2^{(k)} \circ \varphi_2^{(k)}(v).$$

2. (a) If $v' \twoheadrightarrow v$ in $\mathrm{Lea}_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_1^{(k)}(v) = \varphi(v) \geq \varphi(v') = \varphi_2^{(k)}(v')$.
   (b) If $v' \twoheadrightarrow v$ in $\mathrm{Roo}_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_2^{(k)}(v) = \varphi(v) - k \geq \varphi(v') - k = \varphi_2^{(k)}(v')$.

Hence, there is a bijection:

$$\left\{ \begin{array}{rcl} \mathbb{S}^{\tau}_{(F,\sigma)} \times \{0,\dots,|F|_v\} & \rightarrow & \displaystyle\bigsqcup_{\boldsymbol{v} \models V(F)} \mathbb{S}^{\tau_1^{(k)}}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1^{(k)})} \times \mathbb{S}^{\tau_2^{(k)}}_{(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2^{(k)})} \\ (\varphi,k) & \mapsto & \left(\varphi_1^{(k)}, \varphi_2^{(k)}\right). \end{array} \right.$$

Finally,

$$\begin{aligned} & \Delta_{\mathbf{WQSym}^*} \circ \Phi((F,\sigma)) \\ = & \sum_{\tau \in \mathrm{Surj}_{|F|_v}} \sum_{0 \leq k \leq n} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau_1^{(k)} \otimes \tau_2^{(k)} \\ = & \sum_{\boldsymbol{v} \models V(F)} \sum_{\tau_1 \in \mathrm{Surj}_{|\mathrm{Lea}_{\boldsymbol{v}}(F)|_v}} \sum_{\tau_2 \in \mathrm{Surj}_{|\mathrm{Roo}_{\boldsymbol{v}}(F)|_v}} \mathrm{card}(\mathbb{S}^{\tau_1}_{(\mathrm{Lea}_{\boldsymbol{v}}(F),\sigma_1^{(k)})})\ \tau_1 \otimes \mathrm{card}(\mathbb{S}^{\tau_2}_{(\mathrm{Roo}_{\boldsymbol{v}}(F),\sigma_2^{(k)})})\ \tau_2 \\ = & (\Phi \otimes \Phi) \circ \Delta_{\mathbf{H}_{po}}. \end{aligned}$$

So Φ is a coalgebra morphism. □

Theorem 10 *The restriction of Φ defined in formula (7) to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras.*

Remark. The restriction of Φ to $\mathbf{H}_{hpo}$ is not bijective (by comparing the dimensions of $\mathbf{H}_{hpo}$ and $\mathbf{WQSym}^*$ in small degrees).

Proof. We introduce a lexicographic order on the words with letters $\in \mathbb{N}^*$. Let $u = (u_1 \dots u_k)$ and $v = (v_1 \dots v_l)$ be two words. Then

- if $u_k = v_k, u_{k-1} = v_{k-1}, \dots, u_{i+1} = v_{i+1}$ and $u_i > v_i$ (resp. $u_i < v_i$) with $i \in \{1,\dots,\min(k,l)\}$, then $u > v$ (resp. $u < v$),
- if $u_i = v_i$ for all $i \in \{1,\dots,\min(k,l)\}$ and if $k > l$ (resp. $k < l$) then $u > v$ (resp. $u < v$).

For example,

$$(541) < (22), \quad (433) < (533), \quad (5362) < (72), \quad (8225) < (1327), \quad (215) < (1215).$$

If u and v are two words, we denote by uv the concatenation of u and v.

In this proof, if (F,σ) is a preordered forest, we consider F as a decorated forest where the vertices are decorated by integers. Consider

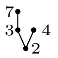
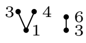
$$\mathbb{F} = \{(F,d) \mid F \in \mathbb{F}_{\mathbf{H}_{CK}}, d : V(F) \rightarrow \mathbb{N}^* \text{ such that if } v \rightarrow w \text{ then } d(v) > d(w)\}$$

the set of forests with their vertices decorated by nonzero integers and with an increasing condition.

Let $(F,d) \in \mathbb{F}$ be a forest of vertex degree n and if $u = (u_1 \dots u_n)$ is a word of length n with $u_i \in \mathbb{N}^*$. In the same way that definition 8, we define $\mathbb{S}^u_{(F,d)}$ as the set of bijective maps $\varphi : V(F) \to \{1,\dots,n\}$ such that:

1. if $v \in V(F)$, then $d(v) = u_{\varphi(v)}$,
2. if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.

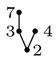
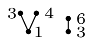
For example,

- if $(F,d) =$  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are $(7342), (7432), (4732)$.
- if $(F,d) =$  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are

$$\begin{array}{l}(43163),(43613),(46313),(64313),(43631),(46331),(64331),(63431),\\(34163),(34613),(36413),(63413),(34631),(36431),(36341),(63341).\end{array}$$

Let (F,d) be a forest of $\mathbb{F}$. Then we denote by

$$m((F,d)) = \max\left(\{u \mid \mathbb{S}^u_{(F,d)} \neq \emptyset\}\right).$$

For example, for $(F,d) =$  $\in \mathbb{F}$, $m((F,d)) = (7342)$ and for $(F,d) =$  $\in \mathbb{F}$, $m((F,d)) = (34163)$.

If $(F,d) \in \mathbb{F}$ is the empty tree, $m((F,\sigma)) = 1$. Let $(F,d) \in \mathbb{F}$ be a nonempty tree of vertex degree n. We denote by (G,d') the forest of $\mathbb{F}$ obtained by deleting the root of F. Then, if $m((F,d)) = (u_1 \dots u_n)$, we have $m((G,d')) = (u_1 \dots u_{n-1})$ and $u_n = d(R_F)$ the decoration of the root of F. Let (F,d) be a forest of vertex degree n, (F,d) is the disjoint union of trees with their vertex decorated by nonzero integers $(F_1,d_1),\dots,(F_k,d_k)$ ordered such that $m((F_1,d_1)) \leq \dots \leq m((F_k,d_k))$. Then $m((F,d)) = m((F_1,d_1))\dots m((F_k,d_k))$:

- By definition, $\mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)} \neq \emptyset$ and if $\varphi_i \in \mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)}$ then $\varphi : V(F) \to \{1,\dots,n\}$, defined for all $1 \leq i \leq k$ and $v \in V(F_i)$ by $\varphi(v) = \varphi_i(v)$, is an element of $\mathbb{S}^{m((F_1,d_1))\dots m((F_k,d_k))}_{(F,d)}$ and $\mathbb{S}^{m((F_1,d_1))\dots m((F_k,d_k))}_{(F,d)} \neq \emptyset$. So $m((F,d)) \geq m((F_1,d_1))\dots m((F_k,d_k))$.
- If $\mathbb{S}^u_{(F,d)} \neq \emptyset$, u is the shuffle of $u_1,\dots,u_k$ such that $\mathbb{S}^{u_i}_{(F_i,d_i)} \neq \emptyset$ (see the proof of theorem 9). In particular, $u_i \leq m((F_i,d_i))$, so $u \leq m((F_1,d_1))\dots m((F_k,d_k))$ and $m((F,d)) \leq m((F_1,d_1))\dots m((F_k,d_k))$.

Let $(F,d) \in \mathbb{F}$ be a forest of vertex degree n and $m((F,d)) = (u_1 \dots u_n)$. Let i_1 be the smallest index such that $u_1,\dots,u_{i_1-1} > u_{i_1}$ and, for all $j > i_1$, $u_{i_1} \leq u_j$. By construction, there exists a connected component (F_1,d_1) of (F,d) such that $m((F_1,d_1)) = (u_1 \dots u_{i_1})$. Consider the word $(u_{i_1+1} \dots u_n)$. Let $i_2 > i_1$ be the smallest index such that $u_{i_1+1},\dots,u_{i_2-1} > u_{i_2}$ and, for all $j > i_2$, $u_{i_2} \leq u_j$. Then there exists a connected component (F_2,d_2) (different from (F_1,d_1)) such that $m((F_2,d_2)) = (u_{i_1+1} \dots u_{i_2})$. In the same way, we construct $i_3,\dots,i_k$ and $(F_3,d_3),\dots,(F_k,d_k)$. Then we have $m((F,d)) = m((F_1,d_1))\dots m((F_k,d_k))$

Let us prove that m is injective on $\mathbb{F}$ by induction on the vertex degree. If (F,d) is the empty tree, it is obvious. Let (F,d) be a nonempty forest of $\mathbb{F}$ of vertex degree n.

- If (F,d) is a tree, $m((F,d)) = (u_1 \dots u_{n-1}u_n)$ with $u_n = d(R_F)$ the decoration of the root of F. Let (G,d') be the forest of $\mathbb{F}$ obtained by deleting the root of F. Then $m((G,d')) = (u_1 \dots u_{n-1})$. By induction hypothesis, (G,d') is the unique forest of $\mathbb{F}$ such that $m((G,d')) = (u_1 \dots u_{n-1})$. So (F,d) is also the unique forest of $\mathbb{F}$ such that $m((F,\sigma)) = (u_1 \dots u_{n-1}d(R_F))$.
- If (F,d) is not a tree, then (F,d) is the product of trees $(F_1,d_1),\dots,(F_k,d_k)$ of $\mathbb{F}$ ordered such that $m((F_1,d_1)) \leq \dots \leq m((F_k,d_k))$. So $m((F,d)) = m((F_1,d_1))\dots m((F_k,d_k))$. By the induction hypothesis, for all $1 \leq i \leq k$, (F_i,d_i) is the unique tree of $\mathbb{F}$ such that its image by m is $m((F_i,d_i))$. So the product (F,d) of (F_i,d_i)'s is the unique forest of $\mathbb{F}$ such that its image by m is $m((F,d))$.

So m is injective on $\mathbb{F}$. By triangularity, m is injective on $\mathbb{F}_{\mathbf{H}_{hpo}}$ and we deduce that the restriction of Φ to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras. $\square$

4 Hopf algebras of contractions

4.1 Commutative case

In [CEFM11], D. Calaque, K. Ebrahimi-Fard and D. Manchon introduce a new coproduct, called in this paper the contraction coproduct, on the augmentation ideal of $\mathbf{H}_{CK}$ (see also [MS11]).

Definition 11 *Let F be a nonempty rooted forest and $\boldsymbol{e}$ a subset of $E(F)$. Then we denote by*

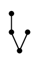
1. $\mathrm{Part}_{\boldsymbol{e}}(F)$ *the subforest of F obtained by keeping all the vertices of F and the edges of $\boldsymbol{e}$,*
2. $\mathrm{Cont}_{\boldsymbol{e}}(F)$ *the forest obtained by contracting each edge of $\boldsymbol{e}$ in F and identifying the two extremities of each edge of $\boldsymbol{e}$.*

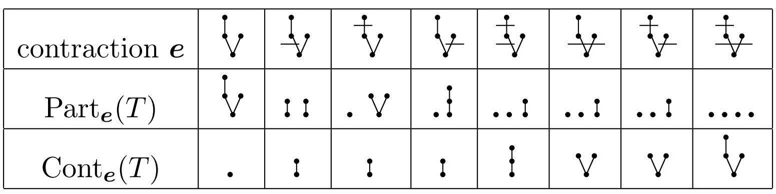
We shall say that $\boldsymbol{e}$ is a contraction of F, $\mathrm{Part}_{\boldsymbol{e}}(F)$ is the partition of F by $\boldsymbol{e}$ and $\mathrm{Cont}_{\boldsymbol{e}}(F)$ is the contracted of F by $\boldsymbol{e}$. Each vertex of $\mathrm{Cont}_{\boldsymbol{e}}(F)$ can be identified to a connected component of $\mathrm{Part}_{\boldsymbol{e}}(F)$.

Remarks.

- If $\boldsymbol{e} = \emptyset$, then $\mathrm{Part}_{\boldsymbol{e}}(F) = \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{|F|_v \times}$ and $\mathrm{Cont}_{\boldsymbol{e}}(F) = F$: this is the *empty contraction* of F.
- If $\boldsymbol{e} = E(F)$, then $\mathrm{Part}_{\boldsymbol{e}}(F) = F$ and $\mathrm{Cont}_{\boldsymbol{e}}(F) = \cdot$: this is the *total contraction* of F.

Notations. We shall write $\boldsymbol{e} \models E(F)$ if $\boldsymbol{e}$ is a contraction of F and $\boldsymbol{e} \mid\models E(F)$ if $\boldsymbol{e}$ is a nonempty, nontotal contraction of F.

Example. Let $T =$  be a rooted tree. Then



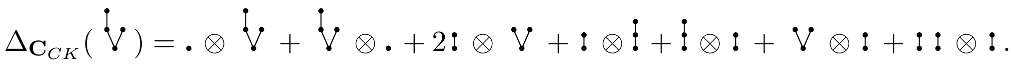
where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

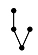
Remarks. Let F be a nonempty rooted forest and $\boldsymbol{e} \models E(F)$.

1. We have the following relation on the vertex degrees:
$$|F|_v = |\mathrm{Cont}_{\boldsymbol{e}}(F)|_v + |\mathrm{Part}_{\boldsymbol{e}}(F)|_v - l(\mathrm{Part}_{\boldsymbol{e}}(F)).$$
2. Note $\overline{\boldsymbol{e}}$ the complementary to $\boldsymbol{e}$ in $E(F)$. Then $E(\mathrm{Part}_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $E(\mathrm{Cont}_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$ and
$$|F|_e = |\mathrm{Cont}_{\boldsymbol{e}}(F)|_e + |\mathrm{Part}_{\boldsymbol{e}}(F)|_e . \tag{8}$$

Let $\mathbf{C}_{CK}$ be the quotient algebra $\mathbf{H}_{CK}/I_{CK}$ where I_{CK} is the ideal spanned by $\bullet - 1$. In others terms, one identifies the unit 1 (for the concatenation) with the tree $\bullet$. We denote by the same way a rooted forest and its class in $\mathbf{C}_{CK}$. Then we define on $\mathbf{C}_{CK}$ a contraction coproduct on each forest $F \in \mathbf{C}_{CK}$:
$$\begin{aligned}\Delta_{\mathbf{C}_{CK}}(F) &= \sum_{\boldsymbol{e} \models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F),\\ &= F \otimes \bullet + \bullet \otimes F + \sum_{\boldsymbol{e} \Vert\!\models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F).\end{aligned}$$
In particular, $\Delta_{\mathbf{C}_{CK}}(\bullet) = \bullet \otimes \bullet$.

Example.



We define an algebra morphism ε:
$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{C}_{CK} & \to & \mathbb{K} \\ F \text{ forest} & \mapsto & \delta_{F,\bullet} . \end{array} \right.$$
Then $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is a commutative Hopf algebra graded by the number of edges. $\mathbf{C}_{CK}$ is non-cocommutative (see for example the coproduct of ).

Remark. We define inductively:
$$\Delta^{(0)}_{\mathbf{C}_{CK}} = Id, \qquad \Delta^{(1)}_{\mathbf{C}_{CK}} = \Delta_{\mathbf{C}_{CK}}, \qquad \Delta^{(k)}_{\mathbf{C}_{CK}} = (\Delta_{\mathbf{C}_{CK}} \otimes Id^{\otimes(k-1)}) \circ \Delta^{(k-1)}_{\mathbf{C}_{CK}}.$$
For all $k \in \mathbb{N}$, $\Delta^{(k)}_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}^{\otimes(k+1)}$. If F is a rooted forest with n edges, there are $(k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$:

- If $k = 0$, this is obvious.
- If $k > 0$, we have $\binom{n}{l}$ tensors $F^{(1)} \otimes F^{(2)}$ in $\Delta_{\mathbf{C}_{CK}}(F)$ such that the left term $F^{(1)}$ have l edges. By the induction hypothesis, there are k^l terms in $\Delta^{(k-1)}_{\mathbf{C}_{CK}}(F^{(1)})$. So there are $\displaystyle\sum_{0 \le l \le n} \binom{n}{l} k^l = (k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$.

We give the first numbers of trees $t_n^{\mathbf{C}_{CK}}$ and forests $f_n^{\mathbf{C}_{CK}}$:

n	0	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842
$f_n^{\mathbf{C}_{CK}}$	1	1	3	7	19	47	127	330	889	2378	6450

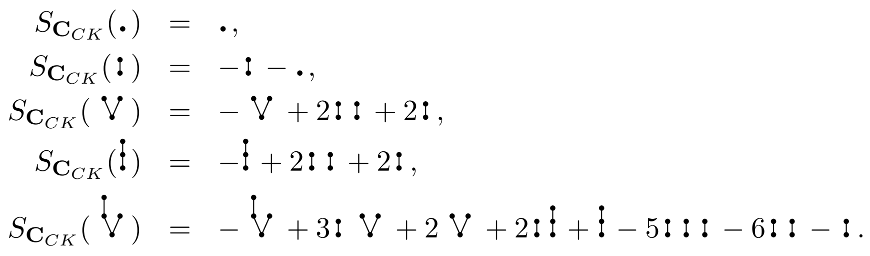
The first sequence is the sequence A000081 in [Slo].

We recall a combinatorial description of the antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}$ (see [CEFM11]):

Proposition 12 *The antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}$ of the Hopf algebra $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is given (recursively with respect to number of edges) by the following formulas: for all forest $F \in \mathbf{C}_{CK}$,*

$$\begin{aligned} S_{\mathbf{C}_{CK}}(F) &= -F - \sum_{\boldsymbol{e} \| \models E(F)} S_{\mathbf{C}_{CK}}(\mathrm{Part}_{\boldsymbol{e}}(F))\, \mathrm{Cont}_{\boldsymbol{e}}(F) \\ &= -F - \sum_{\boldsymbol{e} \| \models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) S_{\mathbf{C}_{CK}}(\mathrm{Cont}_{\boldsymbol{e}}(F)). \end{aligned}$$

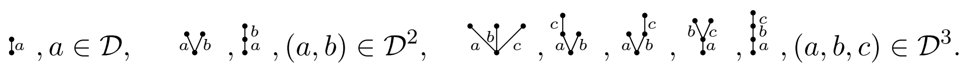
Examples.



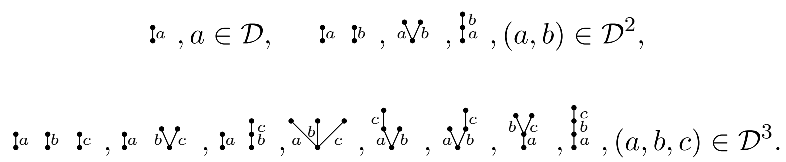
We now give a decorated version of $\mathbf{C}_{CK}$. Let $\mathcal{D}$ be a nonempty set. A rooted forest with their edges decorated by $\mathcal{D}$ is a pair (F, d) where F is a forest of $\mathbf{C}_{CK}$ and $d : E(F) \to \mathcal{D}$ is a map. We denote by $\mathbf{C}_{CK}^{\mathcal{D}}$ the $\mathbb{K}$-vector space spanned by rooted forests with edges decorated by $\mathcal{D}$.

Examples.

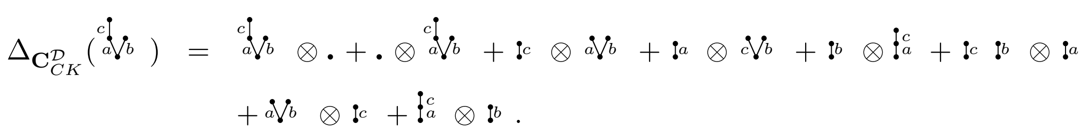
1. Rooted trees decorated by $\mathcal{D}$ with edge degree smaller than 3:



2. Rooted forests decorated by $\mathcal{D}$ with edge degree smaller than 3:



If $F \in \mathbf{C}_{CK}^{\mathcal{D}}$ $\boldsymbol{e} \models E(F)$, then $\mathrm{Part}_{\boldsymbol{e}}(F)$ and $\mathrm{Cont}_{\boldsymbol{e}}(F)$ are naturally rooted forests with their edges decorated by $\mathcal{D}$: we keep the decoration of each edges. The vector space $\mathbf{C}_{CK}^{\mathcal{D}}$ is a Hopf algebra. Its product is given by the concatenation and its coproduct is the contraction coproduct. For example: if $(a, b, c) \in \mathcal{D}^3$,



Notation. The set of nonempty trees of $\mathbf{C}_{CK}$ (that is to say, with at least one edge) will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}$. The set of nonempty trees with their edges decorated by $\mathcal{D}$ of $\mathbf{C}_{CK}^{\mathcal{D}}$ will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}^{\mathcal{D}}$.

4.2 Insertion operations

Let $\mathbf{T}^{\mathcal{D}}_{CK}$ be the $\mathbb{K}$-vector space having for basis $\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$. In this section, we prove that $\mathbf{T}^{\mathcal{D}}_{CK}$ is equiped with two operations $\curlyvee$ and $\triangleright$ such that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ is a commutative prelie algebra.

Definition 13 *1. A commutative prelie algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $\curlyvee, \triangleright : A \otimes A \to A$ such that $x \curlyvee y = y \curlyvee x$ for all $x, y \in A$ (that is to say, $\curlyvee$ is commutative) and satisfying the following relations : for all $x, y, z \in A$,*

$$\left\{\begin{array}{l} (x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z), \\ x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z = y \triangleright (x \triangleright z) - (y \triangleright x) \triangleright z, \\ x \triangleright (y \curlyvee z) = (x \triangleright y) \curlyvee z + (x \triangleright z) \curlyvee y. \end{array}\right. \tag{9}$$

In other words, $(A, \curlyvee, \triangleright)$ is a commutative prelie algebra if $(A, \curlyvee)$ is a commutative algebra and $(A, \triangleright)$ is a left prelie algebra with a relationship between $\curlyvee$ and $\triangleright$.

2. The commutative prelie operad, denoted by $\mathcal{ComPreLie}$, is the operad such that $\mathcal{ComPreLie}$-algebras are commutative prelie algebras.

Remark. From this definition, it is clear that the operad $\mathcal{ComPreLie}$ is binary and quadratic (see [LV12] for a definition).

Notations.

1. Let $T \in \mathbb{T}_{\mathbf{C}_{CK}}$ be a tree with at least one edge. We denote by $V^*(T) = V(T) \setminus \{R_T\}$ the set of vertices of T different from the root of T.

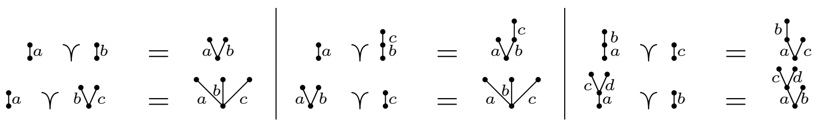
2. Let $T_1, T_2 \in \mathbb{T}_{\mathbf{C}_{CK}}$ and $v \in V(T_2)$. Then $T_1 \circ_v T_2$ is the tree obtained by identifying the root R_{T_1} of T_1 and the vertex v of T_2.

We define two $\mathbb{K}$-linear maps $\curlyvee : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$ and $\triangleright : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$ as follows: if $T_1, T_2 \in \mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$,

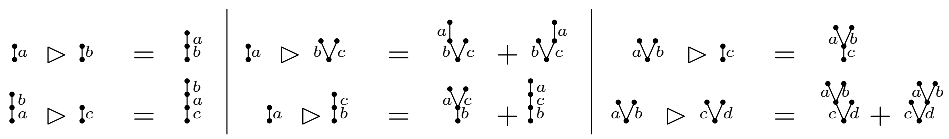
$$\begin{array}{rcl} T_1 \curlyvee T_2 & = & T_1 \circ_{R_{T_2}} T_2, \\ T_1 \triangleright T_2 & = & \displaystyle\sum_{s \in V^*(T_2)} T_1 \circ_s T_2. \end{array}$$

Examples.

1. For the map $\curlyvee : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



2. For the map $\triangleright : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



Proposition 14 $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ *is a $\mathcal{ComPreLie}$-algebra.*

Proof. Let $T_1, T_2, T_3 \in \mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$. Then

$$T_1 \curlyvee T_2 = T_1 \circ_{R_{T_2}} T_2 = T_2 \circ_{R_{T_1}} T_1 = T_2 \curlyvee T_1.$$

Moreover,

$$(T_1 \curlyvee T_2) \curlyvee T_3 = (T_1 \circ_{R_{T_2}} T_2) \circ_{R_{T_3}} T_3 = T_1 \circ_{R_{(T_2 \circ_{R_{T_3}} T_3)}} (T_2 \circ_{R_{T_3}} T_3) = T_1 \curlyvee (T_2 \curlyvee T_3).$$

Therefore $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee)$ is a commutative algebra.

$$\begin{aligned}
T_1 \rhd (T_2 \rhd T_3) &= \sum_{\substack{v \in V^*(T_3)\\ w \in V^*(T_2) \cup V^*(T_3)}} T_1 \circ_w (T_2 \circ_v T_3)\\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} T_1 \circ_w (T_2 \circ_v T_3) + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3)\\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} (T_1 \circ_w T_2) \circ_v T_3 + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3)\\
&= (T_1 \rhd T_2) \rhd T_3 + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3).
\end{aligned}$$

So

$$\begin{aligned}
T_1 \rhd (T_2 \rhd T_3) - (T_1 \rhd T_2) \rhd T_3 &= \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3)\\
&= \sum_{v, w \in V^*(T_3)} T_2 \circ_v (T_1 \circ_w T_3)\\
&= T_2 \rhd (T_1 \rhd T_3) - (T_2 \rhd T_1) \rhd T_3.
\end{aligned}$$

Therefore, $(\mathbf{T}^{\mathcal{D}}_{CK}, \rhd)$ is a left prelie algebra.

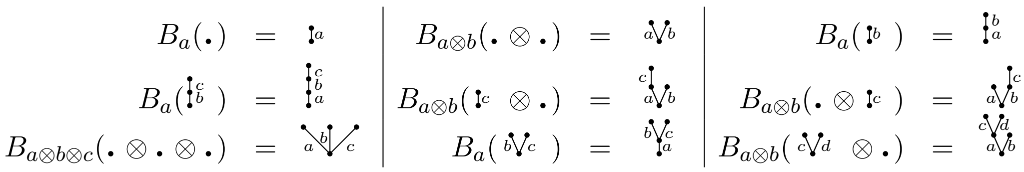
It remains to prove the last relation of (9):

$$\begin{aligned}
T_1 \rhd (T_2 \curlyvee T_3) &= \sum_{v \in V^*(T_2 \circ_{R_{T_3}} T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3)\\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3)\\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_3 \circ_{R_{T_2}} T_2)\\
&= \left(\sum_{v \in V^*(T_2)} T_1 \circ_v T_2 \right) \circ_{R_{T_3}} T_3 + \left(\sum_{v \in V^*(T_3)} T_1 \circ_v T_3 \right) \circ_{R_{T_2}} T_2\\
&= (T_1 \rhd T_2) \curlyvee T_3 + (T_1 \rhd T_3) \curlyvee T_2.
\end{aligned}$$

$\square$

Theorem 15 *$(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ is generated as a $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra by $\bullet_d$, $d \in \mathcal{D}$.*

Notation. To prove the previous proposition, we introduce a notation. Let $T_1, \ldots, T_k$ are trees (possibly empty) of $\mathbf{C}^{\mathcal{D}}_{CK}$ and $d_1, \ldots, d_k \in \mathcal{D}$. Then $B_{d_1 \otimes \ldots \otimes d_k}(T_1 \otimes \ldots \otimes T_k)$ is the tree obtained by grafting each T_i on a common root with an edge decorated by d_i. For examples, if $a, b, c, d \in \mathcal{D}$,



Proof. Let us prove that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ is generated as $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra by $\mid_d$, $d \in \mathcal{D}$ by induction on the edge degree n. If $n = 1$, this is obvious. Let $T \in \mathbf{T}^{\mathcal{D}}_{CK}$ be a tree of edge degree $n \geq 2$. Let k be an integer such that $T = B_{d_1 \otimes \ldots \otimes d_k}(T_1 \otimes \ldots \otimes T_k)$ with $d_1, \ldots, d_k \in \mathcal{D}$ and $T_1, \ldots, T_k$ trees (possibly empty) of $\mathbf{C}^{\mathcal{D}}_{CK}$. Then:

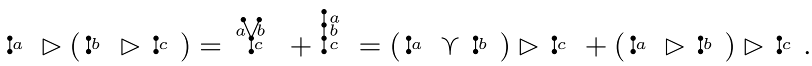
1. If $k = 1$, $T = B_{d_1}(T_1)$ with $|T_1|_e = n - 1 \geq 1$. By induction hypothesis, T_1 can be constructed from trees $\mid_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$. So $T = T_1 \triangleright \mid_{d_1}$ can be also constructed from trees $\mid_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$.

2. Suppose that $k \geq 2$. Then, for all i, $1 \leq |B_{d_i}(T_i)|_e \leq n - 1$. By induction hypothesis, the trees $B_{d_i}(T_i)$ can be constructed from trees $\mid_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$. So $T = B_{d_1}(T_1) \curlyvee \ldots \curlyvee B_{d_k}(T_k)$ can be also constructed from trees $\mid_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$.

We conclude with the induction principle. $\square$

Remarks.

1. $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ is not the free $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra generated by $\mid_d$, $d \in \mathcal{D}$. For example,



2. A description of the free $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra is given in [Foi13].

4.3 Noncommutative case

We give a noncommutative version of $\mathbf{C}_{CK}$. To do this, we work on the algebra $\mathbf{H}_{po}$.

Definition 16 *Let (F, σ^F) be a nonempty preordered forest. In particular, F is a nonempty rooted forest. Let $\boldsymbol{e}$ be a contraction of F, $\mathrm{Part}_{\boldsymbol{e}}(F)$ the partition of F by $\boldsymbol{e}$ and $\mathrm{Cont}_{\boldsymbol{e}}(F)$ the contracted of F by $\boldsymbol{e}$ (see definition 11). Then:*

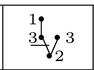
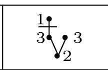
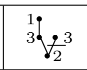
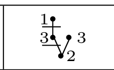
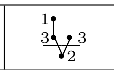
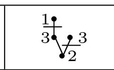
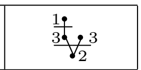
1. *$\mathrm{Part}_{\boldsymbol{e}}(F)$ is a preordered forest $(\mathrm{Part}_{\boldsymbol{e}}(F), \sigma^P)$ where $\sigma^P : v \in V(Part_{\boldsymbol{e}}(F)) \mapsto \sigma^F(v)$. In other words, we keep the initial preorder of the vertices of F in $\mathrm{Part}_{\boldsymbol{e}}(F)$.*

2. *$\mathrm{Cont}_{\boldsymbol{e}}(F)$ is also a preordered forest $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C)$ where $\sigma^C : V(\mathrm{Cont}_{\boldsymbol{e}}(F)) \rightarrow \{1, \ldots, p\}$ is the surjection ($p \leq |\mathrm{Cont}_{\boldsymbol{e}}(F)|_v$) such that if A, B are two connected components of $\mathrm{Part}_{\boldsymbol{e}}(F)$, if a (resp. b) is the vertex obtained by contracting A (resp. B) in F, then*

$$\begin{cases} \sigma^F(R_A) < \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) < \sigma^C(b), \\ \sigma^F(R_A) = \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) = \sigma^C(b), \\ \sigma^F(R_A) > \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) > \sigma^C(b). \end{cases} \tag{10}$$

In other words, we contract each connected component of $\mathrm{Part}_{\boldsymbol{e}}(F)$ to its root and we keep the initial preorder of the roots.

Example. Let $T = {}^{\overset{1}{3}}\mathrm{V}^{3}_{2}$ be a preordered tree. Then

contraction $\boldsymbol{e}$	${}^{\overset{1}{3}}\mathrm{V}^{3}_{2}$							
$\mathrm{Part}_{\boldsymbol{e}}(T)$	${}^{\overset{1}{3}}\mathrm{V}^{3}_{2}$	$\mathbf{l}^{3}_{2}\,\mathbf{l}^{1}_{3}$	$\bullet_1\ {}^{3}\mathrm{V}^{3}_{2}$	$\mathbf{l}^{1}_{\substack{3\\2}}\ \bullet_3$	$\bullet_1\ \mathbf{l}^{3}_{2}\ \bullet_3$	$\bullet_2\ \bullet_3\ \mathbf{l}^{1}_{3}$	$\bullet_1\ \mathbf{l}^{3}_{2}\ \bullet_3$	$\bullet_1\ \bullet_2\ \bullet_3\ \bullet_3$
$\mathrm{Cont}_{\boldsymbol{e}}(T)$	$\bullet_1$	$\mathbf{l}^{2}_{1}$	$\mathbf{l}^{1}_{2}$	$\mathbf{l}^{2}_{1}$	$\mathbf{l}^{1}_{\substack{3\\2}}$	${}^{2}\mathrm{V}^{2}_{1}$	${}^{1}\mathrm{V}^{3}_{2}$	${}^{\overset{1}{3}}\mathrm{V}^{3}_{2}$

where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

Let I_{po} be the ideal of $\mathbf{H}_{po}$ generated by the elements $F\bullet_i - \tilde{F}$ with $F\bullet_i \in \mathbf{H}_{po}$ and $\tilde{F}$ the forest contructed from $F\bullet_i$ by deleting the vertex $\bullet_i$ and keeping the same preorder on $V(F)$. For example,

- if $F\bullet_i = {}^{1}\mathrm{V}^{3}_{3}\ \bullet_2$ then $\tilde{F} = {}^{1}\mathrm{V}^{2}_{2}$,
- if $F\bullet_i = {}^{\overset{2}{1}}\mathrm{V}^{3}_{2}\ \bullet_2$ then $\tilde{F} = {}^{\overset{2}{1}}\mathrm{V}^{3}_{2}$.

Let $\mathbf{C}_{po}$ be the quotient algebra $\mathbf{H}_{po}/I_{po}$. So one identifies the unit 1 (for the concatenation) with the tree $\bullet_1$. Note that $\mathbf{C}_{po}$ is a graded algebra by the number of edges. We denote by the same way a forest and its class in $\mathbf{C}_{po}$. We define on $\mathbf{C}_{po}$ a contraction coproduct on each preordered forest $F \in \mathbf{C}_{po}$:

$$\begin{aligned}
\Delta_{\mathbf{C}_{po}}(F) &= \sum_{\boldsymbol{e} \models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F), \\
&= F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_{\boldsymbol{e} \| \models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F).
\end{aligned}$$

Examples.

$$\begin{aligned}
\Delta_{\mathbf{C}_{po}}(\bullet_1) &= \bullet_1 \otimes \bullet_1 \\
\Delta_{\mathbf{C}_{po}}(\mathbf{l}^{1}_{2}) &= \mathbf{l}^{1}_{2} \otimes \bullet_1 + \bullet_1 \otimes \mathbf{l}^{1}_{2} \\
\Delta_{\mathbf{C}_{po}}({}^{1}\mathrm{V}^{2}_{2}) &= {}^{1}\mathrm{V}^{2}_{2} \otimes \bullet_1 + \bullet_1 \otimes {}^{1}\mathrm{V}^{2}_{2} + \mathbf{l}^{1}_{2} \otimes \mathbf{l}^{1}_{1} + \mathbf{l}^{1}_{1} \otimes \mathbf{l}^{1}_{2} \\
\Delta_{\mathbf{C}_{po}}(\mathbf{l}^{4}_{2}\,\mathbf{l}^{1}_{3}) &= \mathbf{l}^{4}_{2}\,\mathbf{l}^{1}_{3} \otimes \bullet_1 + \bullet_1 \otimes \mathbf{l}^{4}_{2}\,\mathbf{l}^{1}_{3} + \mathbf{l}^{2}_{1} \otimes \mathbf{l}^{1}_{2} + \mathbf{l}^{1}_{2} \otimes \mathbf{l}^{2}_{1} \\
\Delta_{\mathbf{C}_{po}}({}^{\overset{1}{3}}\mathrm{V}^{3}_{2}) &= {}^{\overset{1}{3}}\mathrm{V}^{3}_{2} \otimes \bullet_1 + \bullet_1 \otimes {}^{\overset{1}{3}}\mathrm{V}^{3}_{2} + \mathbf{l}^{3}_{2}\,\mathbf{l}^{1}_{3} \otimes \mathbf{l}^{2}_{1} + {}^{2}\mathrm{V}^{2}_{1} \otimes \mathbf{l}^{1}_{2} + \mathbf{l}^{1}_{\substack{3\\2}} \otimes \mathbf{l}^{2}_{1} + \mathbf{l}^{2}_{1} \otimes \mathbf{l}^{1}_{\substack{3\\2}} \\
&\quad + \mathbf{l}^{1}_{2} \otimes {}^{2}\mathrm{V}^{2}_{1} + \mathbf{l}^{2}_{1} \otimes {}^{1}\mathrm{V}^{3}_{2} \\
\Delta_{\mathbf{C}_{po}}({}^{2}\mathrm{V}^{5}_{3}\,\mathbf{l}^{1}_{4}) &= {}^{2}\mathrm{V}^{5}_{3}\,\mathbf{l}^{1}_{4} \otimes \bullet_1 + \bullet_1 \otimes {}^{2}\mathrm{V}^{5}_{3}\,\mathbf{l}^{1}_{4} + \mathbf{l}^{1}_{2} \otimes \mathbf{l}^{4}_{2}\,\mathbf{l}^{1}_{3} + \mathbf{l}^{2}_{1} \otimes \mathbf{l}^{2}_{3}\,\mathbf{l}^{1}_{4} + \mathbf{l}^{1}_{2} \otimes {}^{1}\mathrm{V}^{3}_{2} \\
&\quad + \mathbf{l}^{2}_{3}\,\mathbf{l}^{1}_{4} \otimes \mathbf{l}^{2}_{1} + \mathbf{l}^{4}_{2}\,\mathbf{l}^{1}_{3} \otimes \mathbf{l}^{1}_{2} + {}^{1}\mathrm{V}^{3}_{2} \otimes \mathbf{l}^{1}_{2}
\end{aligned}$$

Remark. $\Delta_{\mathbf{C}_{po}}$ is non-cocommutative (see for example the coproduct of ${}^{\overset{1}{3}}\mathrm{V}^{3}_{2}$). In particular, if T is a preordered tree and $\boldsymbol{e} \models E(T)$, $\mathrm{Cont}_{\boldsymbol{e}}(T)$ is a preordered tree and $\mathrm{Part}_{\boldsymbol{e}}(T)$ can be disconnected. The second component of the coproduct is linear: a tree instead of a polynomial in trees. This is a right combinatorial Hopf algebra (see [LR10]).

Proposition 17 *1. $\Delta_{\mathbf{C}_{po}}$ is a graded algebra morphism.*

2. $\Delta_{\mathbf{C}_{po}}$ is coassociative.

Proof.

1. Let F, G be two preordered forests. Then

$$\begin{aligned}
\Delta_{\mathbf{C}_{po}}(FG) &= \sum_{\boldsymbol{e}\models E(FG)} \mathrm{Part}_{\boldsymbol{e}}(FG) \otimes \mathrm{Cont}_{\boldsymbol{e}}(FG) \\
&= \sum_{\boldsymbol{e}\models E(F), \boldsymbol{f}\models E(G)} (\mathrm{Part}_{\boldsymbol{e}}(F)\,\mathrm{Part}_{\boldsymbol{f}}(G)) \otimes (\mathrm{Cont}_{\boldsymbol{e}}(F)\,\mathrm{Cont}_{\boldsymbol{f}}(G)) \\
&= \left(\sum_{\boldsymbol{e}\models E(F)} \mathrm{Part}_{\boldsymbol{e}}(F) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F)\right)\left(\sum_{\boldsymbol{f}\models E(G)} \mathrm{Part}_{\boldsymbol{f}}(G) \otimes \mathrm{Cont}_{\boldsymbol{f}}(G)\right) \\
&= \Delta_{\mathbf{C}_{po}}(F)\Delta_{\mathbf{C}_{po}}(G),
\end{aligned}$$

   and $\Delta_{\mathbf{C}_{po}}$ is an algebra morphism. It is a graded algebra morphism with (8).

2. Let F be a nonempty preordered forest. Then

$$\begin{aligned}
&(\Delta_{\mathbf{C}_{po}} \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) \\
=& \sum_{\boldsymbol{e}\models E(F)} \Delta_{\mathbf{C}_{po}}(\mathrm{Part}_{\boldsymbol{e}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F) \\
=& \sum_{\boldsymbol{e}\models E(F)} \sum_{\boldsymbol{f}\models E(\mathrm{Part}_{\boldsymbol{e}}(F))} \mathrm{Part}_{\boldsymbol{f}}(\mathrm{Part}_{\boldsymbol{e}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Part}_{\boldsymbol{e}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F) \\
=& \sum_{\boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)} \mathrm{Part}_{\boldsymbol{f}}(F) \otimes \mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Part}_{\boldsymbol{e}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{e}}(F),
\end{aligned}$$

   and

$$\begin{aligned}
&(Id \otimes \Delta_{\mathbf{C}_{po}}) \circ \Delta_{\mathbf{C}_{po}}(F) \\
=& \sum_{\boldsymbol{f}\models E(F)} \mathrm{Part}_{\boldsymbol{f}}(F) \otimes \Delta_{\mathbf{C}_{po}}(\mathrm{Cont}_{\boldsymbol{f}}(F)) \\
=& \sum_{\boldsymbol{f}\models E(F)} \sum_{\boldsymbol{e}\models E(\mathrm{Cont}_{\boldsymbol{f}}(F))} \mathrm{Part}_{\boldsymbol{f}}(F) \otimes \mathrm{Part}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F)) \\
=& \sum_{\boldsymbol{f}\models E(F), \boldsymbol{e}\subseteq\overline{\boldsymbol{f}}} \mathrm{Part}_{\boldsymbol{f}}(F) \otimes \mathrm{Part}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F)) \otimes \mathrm{Cont}_{\boldsymbol{e}\cup\boldsymbol{f}}(F),
\end{aligned}$$

   where to the last equality we use that $E(\mathrm{Cont}_{\boldsymbol{f}}(F)) = \overline{\boldsymbol{f}}$ the complement of $\boldsymbol{f}$ in $E(F)$ and $\mathrm{Cont}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F)) = \mathrm{Cont}_{\boldsymbol{e}\cup\boldsymbol{f}}(F)$.

   Remark that $\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)\}$ and $\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}\}$ are in bijection:

$$\left\{\begin{array}{rcl}
\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)\} & \to & \{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}\} \\
(\boldsymbol{e}, \boldsymbol{f}) & \to & (\boldsymbol{e} \setminus \boldsymbol{f}, \boldsymbol{f}) \\
(\boldsymbol{e} \cup \boldsymbol{f}, \boldsymbol{f}) & \leftarrow & (\boldsymbol{e}, \boldsymbol{f}).
\end{array}\right.$$

   Moreover,

   - in $\mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Part}_{\boldsymbol{e}}(F))$ with $\boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)$: the edges belong to $\boldsymbol{e} \cap \overline{\boldsymbol{f}} = \boldsymbol{e} \setminus \boldsymbol{f}$; the vertices are the connected components of $\mathrm{Part}_{\boldsymbol{e}\cap\boldsymbol{f}}(F) = \mathrm{Part}_{\boldsymbol{f}}(F)$. The preorder on the vertices is given by the preorder on the roots of the connected components of $\mathrm{Part}_{\boldsymbol{f}}(F)$.
   - in $\mathrm{Part}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F))$ with $\boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}$: the edges belong to $\overline{\boldsymbol{f}} \cap \boldsymbol{e} = \boldsymbol{e} \setminus \boldsymbol{f} = \boldsymbol{e}$; the vertices are the connected components of $\mathrm{Part}_{\boldsymbol{f}}(F)$. As in the precedent case, the preorder on the vertices is given by the preorder on the roots of the connected components of $\mathrm{Part}_{\boldsymbol{f}}(F)$.

So $\mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Part}_{\boldsymbol{e}}(F))$ and $\mathrm{Part}_{\boldsymbol{e}}(\mathrm{Cont}_{\boldsymbol{f}}(F))$ are the same forests with the same preorder on the vertices.

Therefore $(\Delta_{\mathbf{C}_{po}} \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) = (Id \otimes \Delta_{\mathbf{C}_{po}}) \circ \Delta_{\mathbf{C}_{po}}(F)$.

$\square$

We now define

$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{C}_{po} & \rightarrow & \mathbb{K} \\ F \text{ forest} & \mapsto & \delta_{F,\bullet_1}. \end{array} \right.$$

ε is an algebra morphism.

Proposition 18 *ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$.*

Proof. Let F be a forest $\in \mathbf{C}_{po}$. We use the Sweedler notation:

$$\Delta_{\mathbf{C}_{po}}(F) = F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}.$$

Then

$$\begin{array}{rcl} (\varepsilon \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) & = & \varepsilon(F)\bullet_1 + \varepsilon(\bullet_1)F + \sum_F \varepsilon(F^{(1)}) \otimes F^{(2)} = F, \\ (Id \otimes \varepsilon) \circ \Delta_{\mathbf{C}_{po}}(F) & = & F\varepsilon(\bullet_1) + \bullet_1\varepsilon(F) + \sum_F F^{(1)}\varepsilon(F^{(2)}) = F. \end{array}$$

Therefore ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$. $\square$

As $(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is graded (by the number of edges) and connected, we have the following theorem:

Theorem 19 *$(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is a Hopf algebra.*

We denote the antipode of the Hopf algebra $\mathbf{C}_{po}$ by $S_{\mathbf{C}_{po}}$. We have the same combinatorial description of $S_{\mathbf{C}_{po}}$ as in the commutative case (see proposition 12). We give some values of $S_{\mathbf{C}_{po}}$:

- In edge degree 0, $S_{\mathbf{C}_{po}}(\bullet_1) = \bullet_1$.
- In edge degree 1, $S_{\mathbf{C}_{po}}(\mathbf{I}^{1}_{1}) = -\mathbf{I}^{1}_{1} - \bullet_1$, $S_{\mathbf{C}_{po}}(\mathbf{I}^{2}_{1}) = -\mathbf{I}^{2}_{1} - \bullet_1$ and $S_{\mathbf{C}_{po}}(\mathbf{I}^{1}_{2}) = -\mathbf{I}^{1}_{2} - \bullet_1$.
- In edge degree 2,

$$\begin{array}{rcl} S_{\mathbf{C}_{po}}({}^{2}\mathbf{V}^{2}_{1}) & = & -{}^{2}\mathbf{V}^{2}_{1} + 2\mathbf{I}^{2}_{1}\mathbf{I}^{4}_{3} + 2\mathbf{I}^{2}_{1}, \\ S_{\mathbf{C}_{po}}({}^{1}\mathbf{V}^{3}_{2}) & = & -{}^{1}\mathbf{V}^{3}_{2} + \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3} + \mathbf{I}^{2}_{1}\mathbf{I}^{3}_{4} + \mathbf{I}^{2}_{1} + \mathbf{I}^{1}_{2}, \\ S_{\mathbf{C}_{po}}(\mathbf{I}^{\substack{1\\2\\1}}) & = & -\mathbf{I}^{\substack{1\\2\\1}} + \mathbf{I}^{2}_{1}\mathbf{I}^{3}_{3} + \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3} + \mathbf{I}^{1}_{1} + \mathbf{I}^{2}_{1}, \\ S_{\mathbf{C}_{po}}(\mathbf{I}^{3}_{2}\mathbf{I}^{1}_{3}) & = & -\mathbf{I}^{3}_{2}\mathbf{I}^{1}_{3} + \mathbf{I}^{2}_{1}\mathbf{I}^{3}_{4} + \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3} + \mathbf{I}^{1}_{2} + \mathbf{I}^{2}_{1}. \end{array}$$

- In edge degree 3,

$$\begin{array}{rcl} S_{\mathbf{C}_{po}}({}^{\substack{1\\3}}\mathbf{V}^{3}_{2}) & = & -{}^{\substack{1\\3}}\mathbf{V}^{3}_{2} + \mathbf{I}^{3}_{2}\mathbf{I}^{1}_{3}\mathbf{I}^{5}_{4} - \mathbf{I}^{2}_{1}\mathbf{I}^{3}_{4}\mathbf{I}^{6}_{5} - \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3}\mathbf{I}^{6}_{5} - \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3} - \mathbf{I}^{2}_{1}\mathbf{I}^{4}_{3} + {}^{2}\mathbf{V}^{2}_{1}\mathbf{I}^{3}_{4} \\ & & -2\mathbf{I}^{2}_{1}\mathbf{I}^{4}_{3}\mathbf{I}^{5}_{6} - 2\mathbf{I}^{2}_{1}\mathbf{I}^{3}_{4} + \mathbf{I}^{\substack{1\\3\\2}}\mathbf{I}^{4}_{3} - \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3}\mathbf{I}^{6}_{5} - \mathbf{I}^{2}_{1}\mathbf{I}^{4}_{3} - \mathbf{I}^{2}_{1}\mathbf{I}^{3}_{4}\mathbf{I}^{6}_{5} - \mathbf{I}^{1}_{2}\mathbf{I}^{4}_{3} \\ & & +\mathbf{I}^{2}_{1}\mathbf{I}^{\substack{3\\5\\4}} + \mathbf{I}^{\substack{1\\3\\2}} + \mathbf{I}^{1}_{2}\,{}^{4}\mathbf{V}^{4}_{3} + {}^{2}\mathbf{V}^{2}_{1} + \mathbf{I}^{2}_{1}\,{}^{3}\mathbf{V}^{5}_{4} + {}^{1}\mathbf{V}^{3}_{2}. \end{array}$$

Let $\mathbf{C}'_{hpo}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-preordered forests, $\mathbf{C}'_o$ be the $\mathbb{K}$-algebra spanned by nonempty ordered forests, $\mathbf{C}'_{ho}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-ordered forests and $\mathbf{C}'_{NCK}$ be the $\mathbb{K}$-algebra spanned by nonempty planar forests. We consider the quotients $\mathbf{C}_{hpo} = \mathbf{C}'_{hpo}/(I_{po} \cap \mathbf{C}'_{hpo})$, $\mathbf{C}_o = \mathbf{C}'_o/(I_{po} \cap \mathbf{C}'_o)$, $\mathbf{C}_{ho} = \mathbf{C}'_{ho}/(I_{po} \cap \mathbf{C}'_{ho})$ and $\mathbf{C}_{NCK} = \mathbf{C}'_{NCK}/(I_{po} \cap \mathbf{C}'_{NCK})$. We have in this case a diagram similar to (6):

$$\begin{tikzcd} \mathbf{C}_{NCK} \arrow[r, hook] & \mathbf{C}_{ho} \arrow[r, hook] \arrow[d, hook] & \mathbf{C}_o \arrow[d, hook] \\ & \mathbf{C}_{hpo} \arrow[r, hook] & \mathbf{C}_{po} \end{tikzcd}$$

where the arrows $\hookrightarrow$ are injective morphisms of algebras. But they are not always morphisms of Hopf algebras (for the contraction coproduct):

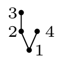
Theorem 20 *1. $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.*

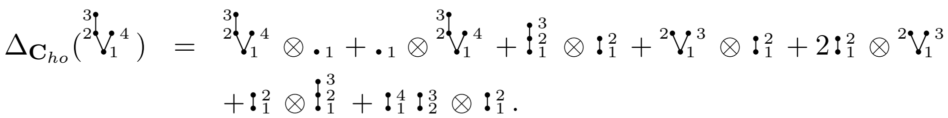
2. $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.

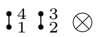
3. $\mathbf{C}_{ho}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_o$ and of the Hopf algebra $\mathbf{C}_{hpo}$.

4. $\mathbf{C}_{NCK}$ is a left comodule of the Hopf algebra $\mathbf{C}_{ho}$.

Notations. We denote by $\Delta_{\mathbf{C}_{hpo}}, \Delta_{\mathbf{C}_o}, \Delta_{\mathbf{C}_{ho}}$ the restrictions of $\Delta_{\mathbf{C}_{po}}$ to $\mathbf{C}_{hpo}, \mathbf{C}_o, \mathbf{C}_{ho}$.

Remark. $\mathbf{C}_{NCK}$ is not a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{ho}$. For example,  $\in \mathbf{C}_{NCK}$ and



Then  $\notin \mathbf{C}_{NCK} \otimes \mathbf{C}_{NCK}$.

Proof.

1. $\mathbf{C}_{hpo}$ is a subalgebra of $\mathbf{C}_{po}$. Let us prove that if $(F, \sigma^F) \in \mathbf{C}_{hpo}$ and $\boldsymbol{e} \models E(F)$ then $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C)$ and $(\mathrm{Part}_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_{hpo}$.

   If $a, b \in V(\mathrm{Part}_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$ then a, b are the vertices of a subtree of $(F, \sigma^F) \in \mathbf{C}_{hpo}$ and $\sigma^F(a) > \sigma^F(b)$. With definition 16, $\sigma^P(a) > \sigma^P(b)$. So $(\mathrm{Part}_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_{hpo}$.

   If $a, b \in V(\mathrm{Cont}_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$, then a and b are the vertices obtained by contracting two connected components A and B of $\mathrm{Part}_{\boldsymbol{e}}(F)$. As $a \twoheadrightarrow b$, $R_A \twoheadrightarrow R_B$ and as $(F, \sigma^F) \in \mathbf{C}_{hpo}$, $\sigma^F(R_A) > \sigma^F(R_B)$. Then, by definition 16, $\sigma^C(a) > \sigma^C(b)$. So $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{hpo}$.

   Therefore if $(F, \sigma^F) \in \mathbf{C}_{hpo}$, $\Delta_{\mathbf{C}_{po}}(F) \in \mathbf{C}_{hpo} \otimes \mathbf{C}_{hpo}$ and $\mathbf{C}_{hpo}$ is a Hopf subalgebra of $\mathbf{C}_{po}$.

2. $\mathbf{C}_o$ is a subalgebra of $\mathbf{C}_{po}$. Let $(F, \sigma^F) \in \mathbf{C}_o$ and $\boldsymbol{e} \models E(F)$. Let us show that $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C)$ and $(\mathrm{Part}_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_o$, that is to say, σ^C and σ^P are bijective.

   By definition 16, σ^P is bijective because we keep the initial order of the vertices of F in $\mathrm{Part}_{\boldsymbol{e}}(F)$. By definition, σ^C is a surjection. Let $a, b \in V(\mathrm{Cont}_{\boldsymbol{e}}(F))$ such that $\sigma^C(a) = \sigma^C(b)$ and A and B be the two connected components of $\mathrm{Part}_{\boldsymbol{e}}(F)$ associated with a and b. With (10), $\sigma^F(R_A) = \sigma^F(R_B)$ and $R_A = R_B$ because σ^F is bijective. So $A = B$, $a = b$ and σ^C is injective.

   Therefore σ^C and σ^P are bijective and $\mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{ho}$.

3. As $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$ and $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$, $\mathbf{C}_{ho} = \mathbf{C}_{hpo} \cap \mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{hpo}$ and $\mathbf{C}_o$.

4. Let us prove that if $(F, \sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{NCK}$. As $\mathbf{C}_{ho}$ is a Hopf algebra, $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{ho}$. So, if $a, b \in V(\mathrm{Cont}_{\boldsymbol{e}}(F))$, such that $a \twoheadrightarrow b$ then $\sigma^C(a) \geq \sigma^C(b)$.

   Moreover, if $a, b, c \in V(\mathrm{Cont}_{\boldsymbol{e}}(F))$ three distinct vertices such that $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b. The vertices a, b and c are obtained by contracting of connected components A, B and C in F. As $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b, $R_A \twoheadrightarrow R_C$, $R_B \twoheadrightarrow R_C$ and R_A is on the left of R_B. As $(F, \sigma^F) \in \mathbf{C}_{NCK}$, $\sigma^F(R_A) < \sigma^F(R_B)$. So $\sigma^C(a) < \sigma^C(b)$.

   Therefore if $(F, \sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(\mathrm{Cont}_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{NCK}$. Consequently, $\Delta_{\mathbf{C}_{ho}}(\mathbf{C}_{NCK}) \subseteq \mathbf{C}_{ho} \otimes \mathbf{C}_{NCK}$.

□

4.4 Formal series

The algebras $\mathbf{C}_{po}$, $\mathbf{C}_{hpo}$, $\mathbf{C}_o$, $\mathbf{C}_{ho}$ and $\mathbf{C}_{NCK}$ are graded by the number of edges.

In the ordered case, we give some values in small degree:

n	1	2	3	4	5	6	7	8
$f_n^{\mathbf{C}_o}$	2	9	76	805	10626	167839	3091768	65127465

These is the sequence A105785 in [Slo].

Let us now study the heap-ordered case. We denote by $f_{n,l}^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edge degree n and of length l, and by $f_n^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edge degree n. In small degree, we have the following values:

$$\begin{cases} f_{0,0}^{\mathbf{C}_{ho}} = f_{1,1}^{\mathbf{C}_{ho}} = 1, & \\ f_{0,l}^{\mathbf{C}_{ho}} = 0 & \text{for all } l \geq 1, \\ f_{1,l}^{\mathbf{C}_{ho}} = 0 & \text{for all } l \neq 1, \\ f_{n,0}^{\mathbf{C}_{ho}} = 0 & \text{for all } n \neq 1. \end{cases}$$

Let n and l be two integers ≥ 1. To obtain a forest $F \in \mathbf{C}_{ho}$ of edge degree n and of length l (so $|F|_v = n + l$), we have two cases :

1. We consider a forest $G \in \mathbf{C}_{ho}$ of edge degree $n-1$ and of length l and we graft the vertex $n+l$ on the vertex i of G. For each forest G, we have $n+l-1$ possibilities.

2. We consider a forest $G \in \mathbf{C}_{ho}$ of edge degree $n-1$ and of length $l-1$. Then, for all $i \in \{1, \ldots, n+l-1\}$, the forest $\tilde{G}\,\mathfrak{I}_{i}^{n+l}$ of edge degree n and of length l is an element of $\mathbf{C}_{ho}$ (where $\tilde{G}$ is the same forest than G where, for all $j \geq i$ the vertex j in G is the vertex $j+1$ in $\tilde{G}$). For each forest G, we have $n+l-1$ possibilities.

So

$$f_{n,l}^{\mathbf{C}_{ho}} = (n+l-1) f_{n-1,l}^{\mathbf{C}_{ho}} + (n+l-1) f_{n-1,l-1}^{\mathbf{C}_{ho}}.$$

We give some values of $f_{n,l}^{\mathbf{C}_{ho}}$ in small degree and small length:

$n\backslash l$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	3	0	0	0
3	0	6	20	15	0	0
4	0	24	130	210	105	0
5	0	120	924	2380	2520	945

Note that $f_{n,1}^{\mathbf{C}_{ho}} = n!$ for all $n \geq 1$. With the formula $f_n^{\mathbf{C}_{ho}} = \sum\limits_{l\geq 0} f_{n,l}^{\mathbf{C}_{ho}}$, we obtain the number of forests of edge degree n. This gives:

n	0	1	2	3	4	5	6
$f_n^{\mathbf{C}_{ho}}$	0	1	5	41	469	6889	123605

This is the sequence A032188 in [Slo].

Remark. Consider the map $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ defined by induction as follows. If $F = 1$, $\varphi(F) = 1$ and if $F = \bullet_1$, $\varphi(F) = (1)$. Let $F \in \mathbf{H}_{ho}$ be a forest of vertice degree n and v the vertex indexed by n. As F is a heap-ordered forest, two cases are possible:

- The vertex v is an isolated vertex. We denote by G the heap-ordered forest obtained by deleting the vertex v of F. Thus $\varphi(G) = \tau'$ is well-defined by induction. Then $\varphi(F)$ is the permutation τ defined by

$$\left\{\begin{array}{rcll} \tau(i) & = & \tau'(i) & \text{if } i \neq n \\ \tau(n) & = & n. & \end{array}\right.$$

- The vertex v is a leaf and we denote by k the index of v' with $v \to v'$. Similarly, we denote by G the heap-ordered forest obtained by deleting the vertex v of F. $\varphi(G) = \tau'$ is well-defined by induction and $\varphi(F)$ is the permutation τ defined by

$$\left\{\begin{array}{rcll} \tau(i) & = & \tau'(i) & \text{if } i \neq k \\ \tau(k) & = & n & \\ \tau(n) & = & \tau'(k). & \end{array}\right.$$

Then $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ is a bijective map. Remark that, if $F \in \mathbb{F}_{\mathbf{H}_{ho}}$, each connected component of F corresponds to one cycle in the writing of $\varphi(F)$ in product of disjoint cycles. Moreover, the restriction of φ to the forests of $\mathbf{C}_{ho}$ is a bijective map with values in the set of permutations without fixed point.

In the planar case, we can obtain the formal series. Let $t_n^{\mathbf{C}_{NCK}}$ be the number of trees $\in \mathbf{C}_{NCK}$ of edge degree n and $f_n^{\mathbf{C}_{NCK}}$ be the number of forests $\in \mathbf{C}_{NCK}$ of edge degree n. We put $T_{\mathbf{C}_{NCK}}(x) = \sum\limits_{k\geq 0} t_k^{\mathbf{C}_{NCK}} x^k$ and $F_{\mathbf{C}_{NCK}}(x) = \sum\limits_{k\geq 0} f_k^{\mathbf{C}_{NCK}} x^k$. Then:

Proposition 21 *The formal series $T_{\mathbf{C}_{NCK}}$ and $F_{\mathbf{C}_{NCK}}$ are given by:*

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1-2x-\sqrt{1-4x}}{2x}, \qquad F_{\mathbf{C}_{NCK}}(x) = \frac{2x}{4x-1+\sqrt{1-4x}}.$$

Proof. With formula (1), we deduce that:

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2x} - 1 = \frac{1-2x-\sqrt{1-4x}}{2x}.$$

$\mathbf{C}_{NCK}$ is freely generated by the trees, therefore

$$F_{\mathbf{C}_{NCK}}(x) = \frac{1}{1-T_{\mathbf{C}_{NCK}}(x)} = \frac{2x}{4x-1+\sqrt{1-4x}}$$

□

Then for all $n \geq 1$ $t_n^{\mathbf{C}_{NCK}} = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, $f_n^{\mathbf{C}_{NCK}} = \binom{2n-1}{n}$ and this gives:

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{NCK}}$	1	2	5	14	42	132	429	1430	4862	16796
$f_n^{\mathbf{C}_{NCK}}$	1	3	10	35	126	462	1716	6435	24310	92378

These are the sequences A000108 and A088218 in [Slo].

5 Hopf algebra morphisms

Recall that the tensor algebra $T(V)$ over a $\mathbb{K}$-vector space V is the tensor module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots$$

equipped with the concatenation.

Dually, the tensor coalgebra $T^c(V)$ over a $\mathbb{K}$-vector space V is the tensor module (as above) equiped with the coassociative coproduct $\Delta_{\mathcal{A}ss}$ called deconcatenation:

$$\Delta_{\mathcal{A}ss}((v_1 \ldots v_n)) = \sum_{i=0}^{n} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n).$$

We will say that a graded bialgebra $\mathbf{H}$ is cofree if, as a graded coalgebra, it is isomorphic to $T^c(\mathrm{Prim}(\mathbf{H}))$ (for more details, see [LR06]).

We give the following useful lemma:

Lemma 22 *Let (A, Δ, ε) be a graded cofree Hopf algebra. Then*

$$\mathrm{Ker}(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta}) = \mathrm{Im}(\tilde{\Delta}).$$

Proof. Indeed, if $x = \sum a_{w,w'} w \otimes w' \in \mathrm{Ker}(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta})$,

$$\sum_{w_1 w_2 = w} a_{w,w'} w_1 \otimes w_2 \otimes w' = \sum_{w'_1 w'_2 = w'} a_{w,w'} w \otimes w'_1 \otimes w'_2.$$

So $a_{w_1 w_2, w_3} = a_{w_1, w_2 w_3}$ for all words w_1, w_2, w_3 different from the unit. We put $b_{ww'} = a_{w,w'}$. Then

$$x = \sum b_w \left(\sum_{w_1 w_2 = w} w_1 \otimes w_2 \right) = \tilde{\Delta}\left(\sum b_w w\right) \in \mathrm{Im}(\tilde{\Delta}).$$

The coassociativity of $\tilde{\Delta}$ implies the other inclusion. □

5.1 From $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map.

Theorem 23 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}_{CK}^{\mathcal{D}} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{H}_{CK}^{\mathcal{D}} \arrow[r, "\Phi"] & \mathbf{Sh}^{\mathcal{D}} \arrow[u, two heads, "\pi"'] \end{tikzcd} \tag{11}$$

is commutative.

Proof. Existence: We define Φ by induction on the number of vertices. We put $\Phi(1) = 1 \otimes 1$ and $\Phi(\bullet_a) = \varphi(\bullet_a)$ for all $a \in \mathcal{D}$. Suppose that Φ is defined for all forest F of vertice degree $< n$ and satisfies the condition $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F)$. Let $F \in \mathbf{H}_{CK}^{\mathcal{D}}$ be a forest of vertice degree n. If $F = F_1F_2$, we put $\Phi(F) = \Phi(F_1)\Phi(F_2)$. Suppose that F is a tree. By induction hypothesis, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F)$ is well-defined. Moreover,

$$\begin{aligned} & (\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}) \circ (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & (\Phi \otimes \Phi \otimes \Phi) \circ (\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes Id_{\mathbf{H}_{CK}^{\mathcal{D}}} - Id_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & 0, \end{aligned}$$

using induction hypothesis in the first equality and the coassociativity in the second equality.

So $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \in \operatorname{Ker}(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$. As $\mathbf{Sh}^{\mathcal{D}}$ is cofree, with lemma 22, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \in \operatorname{Im}(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$ and there exists $w \in \mathbf{Sh}^{\mathcal{D}}$ such that $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w)$. We put $\Phi(F) = w - \pi(w) + \varphi(F)$. Then

$$\begin{aligned} \pi \circ \Phi(F) &= \pi(w) - \pi \circ \pi(w) + \pi \circ \varphi(F) = \varphi(F), \\ \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) - \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\pi(w)) + \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\varphi(F)) \\ &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) \\ &= (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F). \end{aligned}$$

By induction, the result is established.

Uniqueness: Let Φ_1 and Φ_2 be two Hopf algebra morphisms such that the diagram (11) is commutative. Let us prove that $\Phi_1(T) = \Phi_2(T)$ for all tree $T \in \mathbf{H}_{CK}^{\mathcal{D}}$ by induction on the vertice degree of T. If $n = 0$, $\Phi_1(1) = \Phi_2(1) = 1$. If $n = 1$, for $i = 1, 2$, $\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_i(\bullet_a) = (\Phi_i \otimes \Phi_i) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(\bullet_a) = 0$. So $\Phi_i(\bullet_a) \in \operatorname{Vect}(\mathcal{D})$. As the diagram (11) is commutative, $\Phi_1(\bullet_a) = \Phi_2(\bullet_a) = \varphi(\bullet_a)$. Suppose that the result is true in vertice degree $< n$ and let T be a tree of vertice degree n. Using induction hypothesis in the second equality,

$$\begin{aligned} \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_1(T) &= (\Phi_1 \otimes \Phi_1) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T) \\ &= (\Phi_2 \otimes \Phi_2) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T) \\ &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_2(T). \end{aligned}$$

So $\Phi_1(T) - \Phi_2(T) \in \operatorname{Vect}(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}})$ and $\Phi_1(T) - \Phi_2(T) = \pi(\Phi_1(T) - \Phi_2(T)) = \varphi(T) - \varphi(T) = 0$. $\square$

Notation. We consider $F \in \mathbf{H}_{CK}$, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}(\mathrm{Cont}_{\boldsymbol{e}}(F))$ a linear order on $\mathrm{Cont}_{\boldsymbol{e}}(F)$ (see definition 3). For all $i \in \{1, \ldots, |\mathrm{Cont}_{\boldsymbol{e}}(F)|_v\}$, $\sigma^{-1}(i)$ is the connected component of $\mathrm{Part}_{\boldsymbol{e}}(F)$ such that her image by σ is equal to i.

The following proposition gives a combinatorial description of the morphism Φ defined in theorem 23 :

Proposition 24 *Let T be a nonempty tree $\in \mathbf{H}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\sigma \in \mathcal{O}(\mathrm{Cont}_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(|\mathrm{Cont}_{\boldsymbol{e}}(F)|_v)) \ldots \varphi(\sigma^{-1}(1)) \right). \tag{12}$$

Proof. We use the following lemma:

Lemma 25 *Let T be a rooted tree of vertice degree n. We define :*

$$\begin{array}{rcl} \mathbb{E}(T) & = & \{(\boldsymbol{v}, \sigma_1, \sigma_2) \mid \boldsymbol{v} \Vdash V(T), \sigma_1 \in \mathcal{O}(\mathrm{Lea}_{\boldsymbol{v}}(T)), \sigma_2 \in \mathcal{O}(\mathrm{Roo}_{\boldsymbol{v}}(T))\}, \\ \mathbb{F}(T) & = & \{(\sigma, p) \mid \sigma \in \mathcal{O}(T), p \in \{1, \ldots, n-1\}\}. \end{array}$$

Then $\mathbb{E}(T)$ and $\mathbb{F}(T)$ are in bijection.

Proof. We define two maps f and g.
Let f be the map defined by

$$f : \left\{ \begin{array}{rcl} \mathbb{E}(T) & \to & \mathbb{F}(T) \\ (\boldsymbol{v}, \sigma_1, \sigma_2) & \mapsto & (\sigma, |\mathrm{Roo}_{\boldsymbol{v}}(T)|_v) \end{array} \right.$$

where $\sigma : V(T) \to \{1, \ldots, n\}$ is defined by $\sigma(v) = \sigma_2(v)$ for all $v \in V(\mathrm{Roo}_{\boldsymbol{v}}(T))$ and $\sigma(v) = \sigma_1(v) + |\mathrm{Roo}_{\boldsymbol{v}}(T)|_v$ for all $v \in V(\mathrm{Lea}_{\boldsymbol{v}}(T))$. By definition, $\sigma \in \mathcal{O}(T)$.
Let g be the map defined by

$$g : \left\{ \begin{array}{rcl} \mathbb{F}(T) & \to & \mathbb{E}(T) \\ (\sigma, p) & \mapsto & (\boldsymbol{v}, \sigma_1, \sigma_2) \end{array} \right.$$

where

- $\sigma_1 : V(\mathrm{Lea}_{\boldsymbol{v}}(T)) \to \{1, \ldots, |\mathrm{Lea}_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_1(v) = \sigma(v) - |\mathrm{Roo}_{\boldsymbol{v}}(T)|_v$ for all $v \in V(\mathrm{Lea}_{\boldsymbol{v}}(T))$. Then $\sigma_1 \in \mathcal{O}(\mathrm{Lea}_{\boldsymbol{v}}(T))$
- $\sigma_2 : V(\mathrm{Roo}_{\boldsymbol{v}}(T)) \to \{1, \ldots, |\mathrm{Roo}_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_2(v) = \sigma(v)$ for all $v \in V(\mathrm{Roo}_{\boldsymbol{v}}(T))$. Then $\sigma_2 \in \mathcal{O}(\mathrm{Roo}_{\boldsymbol{v}}(T))$
- $\boldsymbol{v}$ is the subset $\left\{v \in \sigma^{-1}(\{k, \ldots, n\}) \mid \text{if } w \in \sigma^{-1}(\{k, \ldots, n\}) \text{ and } v \twoheadrightarrow w \text{ then } v = w\right\}$ of $V(T)$. We have $\boldsymbol{v} \Vdash V(T)$.

So f and g are well-defined. Then we show easily that $f \circ g = Id_{\mathbb{F}(T)}$ and $g \circ f = Id_{\mathbb{E}(T)}$. □

Let us show formula (12) by induction on the number n of vertices. If $n=1$, $T=\bullet_a$ with $a\in\mathcal{D}$. Then $\Phi(\bullet_a)=\varphi(\bullet_a)$ and formula (12) is true. If $n\geq 2$,

$$\begin{aligned}
&\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}\circ\Phi(T)\\
=\;&(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)\\
=\;&\sum_{\boldsymbol{v}\|\models V(T)}\Phi(\mathrm{Lea}_{\boldsymbol{v}}(T))\otimes\Phi(\mathrm{Roo}_{\boldsymbol{v}}(T))\\
=\;&\sum_{\boldsymbol{v}\|\models V(T)}\left(\sum_{\boldsymbol{e}\models E(\mathrm{Lea}_{\boldsymbol{v}}(T))}\left(\sum_{\sigma_1\in\mathcal{O}(\mathrm{Cont}_{\boldsymbol{e}}(\mathrm{Lea}_{\boldsymbol{v}}(T)))}\varphi(\sigma_1^{-1}(|\mathrm{Cont}_{\boldsymbol{e}}(\mathrm{Lea}_{\boldsymbol{v}}(T))|_v))\ldots\varphi(\sigma_1^{-1}(1))\right)\right)\\
&\otimes\left(\sum_{\boldsymbol{f}\models E(\mathrm{Roo}_{\boldsymbol{v}}(T))}\left(\sum_{\sigma_2\in\mathcal{O}(\mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Roo}_{\boldsymbol{v}}(T)))}\varphi(\sigma_2^{-1}(|\mathrm{Cont}_{\boldsymbol{f}}(\mathrm{Roo}_{\boldsymbol{v}}(T))|_v))\ldots\varphi(\sigma_2^{-1}(1))\right)\right)\\
=\;&\sum_{\boldsymbol{e}\models E(T)}\sum_{(\boldsymbol{v},\sigma_1,\sigma_2)\in\mathbb{E}(\mathrm{Cont}_{\boldsymbol{e}}(T))}\varphi(\sigma_1^{-1}(|\mathrm{Lea}_{\boldsymbol{v}}(\mathrm{Cont}_{\boldsymbol{e}}(T))|_v))\ldots\varphi(\sigma_1^{-1}(1))\\
&\qquad\otimes\varphi(\sigma_2^{-1}(|\mathrm{Roo}_{\boldsymbol{v}}(\mathrm{Cont}_{\boldsymbol{e}}(T))|_v))\ldots\varphi(\sigma_2^{-1}(1))\\
=\;&\sum_{\boldsymbol{e}\models E(T)}\sum_{(\sigma,p)\in\mathbb{F}(\mathrm{Cont}_{\boldsymbol{e}}(T))}\varphi(\sigma^{-1}(|\mathrm{Cont}_{\boldsymbol{e}}(T)|_v))\ldots\varphi(\sigma^{-1}(p+1))\otimes\varphi(\sigma^{-1}(p))\ldots\varphi(\sigma^{-1}(1)).
\end{aligned}$$

So

$$\Phi(T)=\sum_{\boldsymbol{e}\models E(T)}\left(\sum_{\sigma\in\mathcal{O}(\mathrm{Cont}_{\boldsymbol{e}}(T))}\varphi(\sigma^{-1}(|\mathrm{Cont}_{\boldsymbol{e}}(F)|_v))\ldots\varphi(\sigma^{-1}(1))\right)$$

and by induction, we have the result. $\square$

Examples.

- In vertex degree 1, $\Phi(\bullet_a)=\varphi(\bullet_a)$.
- In vertex degree 2,

$$\begin{aligned}
\Phi(\mathbf{l}^{b}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_a)+\varphi(\mathbf{l}^{b}_{a})\\
\Phi(\bullet_a\bullet_b) &= \varphi(\bullet_a)\varphi(\bullet_b)+\varphi(\bullet_b)\varphi(\bullet_a).
\end{aligned}$$

- In vertex degree 3,

$$\begin{aligned}
\Phi({}^{b}\mathbf{V}^{c}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a)+\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a)+\varphi(\bullet_b)\varphi(\mathbf{l}^{c}_{a})+\varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a})+\varphi({}^{b}\mathbf{V}^{c}_{a})\\
\Phi(\mathbf{l}^{c}_{b\,a}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a)+\varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a})+\varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_a)+\varphi(\mathbf{l}^{c}_{b\,a})\\
\Phi({}^{c}{}^{b}\mathbf{V}^{d}_{a}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a)+\varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a)+\varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a)\\
&\quad+\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\mathbf{l}^{d}_{a})+\varphi(\bullet_c)\varphi(\bullet_d)\varphi(\mathbf{l}^{b}_{a})+\varphi(\bullet_d)\varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a})+\varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_d)\varphi(\bullet_a)\\
&\quad+\varphi(\bullet_d)\varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_a)+\varphi(\mathbf{l}^{c}_{b})\varphi(\mathbf{l}^{d}_{a})+\varphi(\bullet_c)\varphi({}^{b}\mathbf{V}^{d}_{a})+\varphi(\bullet_d)\varphi(\mathbf{l}^{c}_{b\,a})+\varphi({}^{c}{}^{b}\mathbf{V}^{d}_{a}).
\end{aligned}$$

Particular case. If $\varphi(\bullet_a)=a$ for all $a\in\mathcal{D}$ and $\varphi(T)=0$ if $|T|_v\geq 1$, then this is the particular case of arborification (see [EV04]). For example :

$$\begin{array}{rcl|rcl|rcl}
\Phi(\bullet_a) &=& a & \Phi(\mathbf{l}^{b}_{a}) &=& ba & \Phi(\bullet_a\bullet_b) &=& ab+ba\\
\Phi({}^{b}\mathbf{V}^{c}_{a}) &=& bca+cba & \Phi(\mathbf{l}^{c}_{b\,a}) &=& cba & \Phi({}^{c}{}^{b}\mathbf{V}^{d}_{a}) &=& cbda+cdba+dcba.
\end{array}$$

5.2 From $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 26 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}_{CK}^{\mathcal{D}} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{H}_{CK}^{\mathcal{D}} \arrow[r, "\Phi"] & \mathbf{Csh}^{\mathcal{D}} \arrow[u, two heads, "\pi"'] \end{tikzcd} \tag{13}$$

is commutative.

Proof. Noting that $\mathbf{Csh}^{\mathcal{D}}$ is cofree, this is the same proof as for theorem 23. $\square$

Notation. Let $F \in \mathbf{H}_{CK}$ be a nonempty rooted forest, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}_p(\mathrm{Cont}_{\boldsymbol{e}}(F))$ a linear preorder on $\mathrm{Cont}_{\boldsymbol{e}}(F)$ (see definition 7), $\sigma : V(\mathrm{Cont}_{\boldsymbol{e}}(F)) \to \{1,\ldots,q\}$ surjective. For all $i \in \{1,\ldots,q\}$, $\sigma^{-1}(i)$ is the forest $T_1 \ldots T_n$ of all connected components T_k of $\mathrm{Part}_{\boldsymbol{e}}(F)$ such that $\sigma(T_k) = i$ for all $k \in \{1,\ldots,n\}$. In this case, $\varphi(\sigma^{-1}(i))$ is the element $[\varphi(T_1)\ldots\varphi(T_n)]^{(n)}$.

Now, we give a combinatorial description of the morphism Φ defined in theorem 26:

Proposition 27 *Let T be a nonempty tree $\in \mathbf{H}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\substack{\sigma \in \mathcal{O}_p(\mathrm{Cont}_{\boldsymbol{e}}(T)) \\ \mathrm{Im}(\sigma) = \{1,\ldots,q\}}} \varphi(\sigma^{-1}(q)) \ldots \varphi(\sigma^{-1}(1)) \right). \tag{14}$$

Proof. It suffices to adapt the proof of proposition 24. Note that, if T is a rooted tree and $\boldsymbol{v} \models V(T)$, $\mathrm{Roo}_{\boldsymbol{v}}(T)$ is a tree and $\mathrm{Lea}_{\boldsymbol{v}}(T)$ is a forest. So there are possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T)$. We deduce formula (14). $\square$

Examples.

- In vertex degree 1, $\Phi(\bullet_a) = \varphi(\bullet_a)$.
- In vertex degree 2,

$$\begin{array}{rcl} \Phi(\bullet_a \bullet_b) & = & \varphi(\bullet_a)\varphi(\bullet_b) + \varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_a)\varphi(\bullet_b)] \\ \Phi(\mathbf{l}^{b}_{a}) & = & \varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\mathbf{l}^{b}_{a}). \end{array}$$

- In vertex degree 3,

$$\begin{array}{rcl} \Phi({}^{b}\mathbf{V}^{c}_{a}) & = & \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_c)\varphi(\bullet_b)]\,\varphi(\bullet_a) + \varphi(\bullet_b)\varphi(\mathbf{l}^{c}_{a}) \\ & & +\varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a}) + \varphi({}^{b}\mathbf{V}^{c}_{a}) \\ \Phi(\mathbf{l}^{c}_{b,a}) & = & \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a}) + \varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{l}^{c}_{b,a}) \\ \Phi({}^{c}_{b}\mathbf{V}^{d}_{a}) & = & \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_c)\,[\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a) \\ & & +\,[\varphi(\bullet_c)\varphi(\bullet_d)]\,\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\mathbf{l}^{d}_{a}) \\ & & +\varphi(\bullet_c)\varphi(\bullet_d)\varphi(\mathbf{l}^{b}_{a}) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\mathbf{l}^{b}_{a}) + \varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_d)\varphi(\bullet_a) \\ & & +\,[\varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\mathbf{l}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{l}^{c}_{b})\varphi(\mathbf{l}^{d}_{a}) + \varphi(\bullet_c)\varphi({}^{b}\mathbf{V}^{d}_{a}) \\ & & +\varphi(\bullet_d)\varphi(\mathbf{l}^{c}_{b,a}) + \varphi({}^{c}_{b}\mathbf{V}^{d}_{a}). \end{array}$$

Particular case. If $\varphi(\bullet_a) = a$ for all $a \in \mathcal{D}$ and $\varphi(T) = 0$ if $|T|_v \geq 1$, then this is the particular case of contracting arborification (see [EV04]). For example :

$$\begin{array}{rcl|rcl|rcl} \Phi(\bullet_a) & = & a & \Phi(\mathbf{l}^{b}_{a}) & = & ba & \Phi(\bullet_a \bullet_b) & = & ab + ba + [ab] \\ \Phi({}^{b}\mathbf{V}^{c}_{a}) & = & bca + cba + [bc]a & \Phi(\mathbf{l}^{c}_{b,a}) & = & cba & \Phi({}^{c}\mathbf{Y}^{d}{}^{b}_{a}) & = & dcba + cdba + [cd]ba. \end{array}$$

5.3 From $\mathbf{C}^{\mathcal{D}}_{CK}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map.

Theorem 28 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}^{\mathcal{D}}_{CK} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{C}^{\mathcal{D}}_{CK} \arrow[r, "\Phi"] & \mathbf{Sh}^{\mathcal{D}} \arrow[u, two heads, "\pi"'] \end{tikzcd} \tag{15}$$

is commutative.

Proof. This is the same proof as for theorem 23. □

As in the sections 5.1 and 5.2, we give a combinatorial description of the morphism Φ defined in theorem 28. We need the following definition:

Definition 29 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized partition of F is a k-uplet $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k)$ of subsets of $E(F)$, $1 \leq k \leq |F|_e$, such that:*

1. *$\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ if $i \neq j$ and $\cup_i \boldsymbol{e}_i = E(F)$,*
2. *the edges of any $\boldsymbol{e}_i$ belong to the same connected component of F,*
3. *if v and w are two vertices of $\mathrm{Part}_{\boldsymbol{e}_i}(F)$ and if the shortest path in F between v and w contains an edge $\in \boldsymbol{e}_j$, then $j < i$.*

We shall denote by $\mathcal{P}(F)$ the set of generalized partitions of F.

Proposition 30 *Let F be a nonempty forest $\in \mathbf{C}^{\mathcal{D}}_{CK}$. Then*

$$\Phi(F) = \sum_{(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1}}(F)) \ldots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_k}}(F)). \tag{16}$$

Proof. We use the following lemma:

Lemma 31 *If $F \in \mathbf{C}^{\mathcal{D}}_{CK}$ is a nonempty tree, then the sets*

$$\begin{array}{rcl} \mathbb{E}(F) & = & \{((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p) \mid (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F), 1 \leq p \leq k-1\} \\ \mathbb{F}(F) & = & \{\big(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\big) \mid \boldsymbol{e} \models\!\!\!| E(F), (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(\mathrm{Part}_{\boldsymbol{e}}(F)), \\ & & (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(\mathrm{Cont}_{\boldsymbol{e}}(F))\} \end{array}$$

are in bijection.

Proof. Consider the following two maps:

$$f:\left\{\begin{array}{rcl}\mathbb{E}(F) & \to & \mathbb{F}(F)\\ ((\boldsymbol{e}_1,\dots,\boldsymbol{e}_k),p) & \mapsto & (\cup_{1\le i\le p}\boldsymbol{e}_i,(\boldsymbol{e}_1,\dots,\boldsymbol{e}_p),(\boldsymbol{e}_{p+1},\dots,\boldsymbol{e}_k))\end{array}\right.$$

and

$$g:\left\{\begin{array}{rcl}\mathbb{F}(F) & \to & \mathbb{E}(F)\\ \big(\boldsymbol{e},(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q),(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r)\big) & \mapsto & \big((\boldsymbol{f}_1,\dots,\boldsymbol{f}_q,\boldsymbol{g}_1,\dots,\boldsymbol{g}_r),q\big)\,.\end{array}\right.$$

<u>f is well-defined :</u>

Let $((\boldsymbol{e}_1,\dots,\boldsymbol{e}_k),p)\in\mathbb{E}(F)$. Then $\boldsymbol{e}=\cup_{1\le i\le p}\boldsymbol{e}_i$ is a nonempty nontotal contraction of F.

1. (a) $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_p)$ is a p-uplet of subsets of $E(\mathrm{Part}_{\boldsymbol{e}}(F))=\boldsymbol{e}$. By hypothesis, $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in\mathcal{P}(F)$. So $\boldsymbol{e}_i\neq\emptyset$, $\boldsymbol{e}_i\cap\boldsymbol{e}_j=\emptyset$ and $\cup_{1\le i\le p}\boldsymbol{e}_i=E(\mathrm{Part}_{\boldsymbol{e}}(F))$.
   (b) The edges $\in\boldsymbol{e}_i$, $1\le i\le p$, are the edges of the same connected component of F therefore of $\mathrm{Part}_{\boldsymbol{e}}(F)$ because $\boldsymbol{e}_i\subseteq\boldsymbol{e}$.
   (c) Let v and w be two vertices of $\mathrm{Part}_{\boldsymbol{e}_i}(\mathrm{Part}_{\boldsymbol{e}}(F))=\mathrm{Part}_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i\subseteq\boldsymbol{e}$). If the shortest path in $\mathrm{Part}_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in\boldsymbol{e}_j$, then the shortest path in F between v and w contains also an edge $\in\boldsymbol{e}_j$. As $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in\mathcal{P}(F)$, we have $j<i$.

   So $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_p)\in\mathcal{P}(\mathrm{Part}_{\boldsymbol{e}}(F))$.
2. (a) $(\boldsymbol{e}_{p+1},\dots,\boldsymbol{e}_k)$ is a $(k-p)$-uplet of subsets of $E(\mathrm{Cont}_{\boldsymbol{e}}(F))=\overline{\boldsymbol{e}}$. By hypothesis, $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in\mathcal{P}(F)$. So $\boldsymbol{e}_i\neq\emptyset$, $\boldsymbol{e}_i\cap\boldsymbol{e}_j=\emptyset$ and $\cup_{p+1\le i\le k}\boldsymbol{e}_i=E(\mathrm{Cont}_{\boldsymbol{e}}(F))$.
   (b) The edges $\in\boldsymbol{e}_i$, $p+1\le i\le k$, are the edges of the same connected component of F therefore of $\mathrm{Cont}_{\boldsymbol{e}}(F)$ (we contract in F some connected components).
   (c) Let i be an integer $\in\{p+1,\dots,k\}$ and v and w two vertices of $\mathrm{Part}_{\boldsymbol{e}_i}(\mathrm{Cont}_{\boldsymbol{e}}(F))=\mathrm{Part}_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i\cap\boldsymbol{e}=\emptyset$). If the shortest path in $\mathrm{Cont}_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in\boldsymbol{e}_j$ then the shortest path in F between v and w contains also an edge $\in\boldsymbol{e}_j$. As $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in\mathcal{P}(F)$, we have $j<i$.

   Thus $(\boldsymbol{e}_{p+1},\dots,\boldsymbol{e}_k)\in\mathcal{P}(\mathrm{Cont}_{\boldsymbol{e}}(F))$.

So $f\left(((\boldsymbol{e}_1,\dots,\boldsymbol{e}_k),p)\right)\in\mathbb{F}(F)$.

<u>g is well-defined :</u>

Let $\big(\boldsymbol{e},(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q),(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r)\big)\in\mathbb{F}(F)$. Let us show that $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q,\boldsymbol{g}_1,\dots,\boldsymbol{g}_r)\in\mathcal{P}(F)$.

1. As $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q)\in\mathcal{P}(\mathrm{Part}_{\boldsymbol{e}}(F))$ and $(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r)\in\mathcal{P}(\mathrm{Cont}_{\boldsymbol{e}}(F))$, $\boldsymbol{f}_i\neq\emptyset$, $\boldsymbol{g}_i\neq\emptyset$, $\boldsymbol{f}_i\cap\boldsymbol{f}_j=\emptyset$, $\boldsymbol{g}_i\cap\boldsymbol{g}_j=\emptyset$ and $(\cup_i\boldsymbol{f}_i)\bigcup(\cup_i\boldsymbol{g}_i)=E(\mathrm{Part}_{\boldsymbol{e}}(F))\bigcup E(\mathrm{Cont}_{\boldsymbol{e}}(F))=E(F)$. In addition, as $\boldsymbol{f}_i\subseteq E(\mathrm{Part}_{\boldsymbol{e}}(F))=\boldsymbol{e}$ and $\boldsymbol{g}_j\subseteq E(\mathrm{Cont}_{\boldsymbol{e}}(F))=\overline{\boldsymbol{e}}$, $\boldsymbol{f}_i\cap\boldsymbol{g}_j=\emptyset$.
2. The edges $\in\boldsymbol{f}_i$ are the edges of the same connected component of $\mathrm{Part}_{\boldsymbol{e}}(F)$. As all the trees of the forest $\mathrm{Part}_{\boldsymbol{e}}(F)$ are subtrees of F, the edges $\in\boldsymbol{f}_i$ are the edges of the same connected component of F. Moreover if the edges $\in\boldsymbol{g}_i$ are the edges of the same connected component of $\mathrm{Cont}_{\boldsymbol{e}}(F)$, it is also true in F.
3. (a) Let i be an integer $\in\{1,\dots,q\}$ and v and w two vertices of $\mathrm{Part}_{\boldsymbol{f}_i}(F)$. We have $\boldsymbol{f}_i\subseteq\boldsymbol{e}$ therefore $\mathrm{Part}_{\boldsymbol{f}_i}(F)=\mathrm{Part}_{\boldsymbol{f}_i}(\mathrm{Part}_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains:
      i. an edge $\in\boldsymbol{f}_j$. As $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q)\in\mathcal{P}(\mathrm{Part}_{\boldsymbol{e}}(F))$, $j<i$.
      ii. an edge $\in\boldsymbol{g}_j$. Then the connected component of $\mathrm{Part}_{\boldsymbol{e}}(F)$ containing v and w has an edge $\in\boldsymbol{g}_j$. This is impossible because $E(\mathrm{Part}_{\boldsymbol{e}}(F))=\boldsymbol{e}$ and $\boldsymbol{g}_j\subseteq E(\mathrm{Cont}_{\boldsymbol{e}}(F))=\overline{\boldsymbol{e}}$.

(b) Let i be an integer $\in \{1,\dots,r\}$ and v and w two vertices of $\mathrm{Part}_{\boldsymbol{g}_i}(F)$. $\boldsymbol{g}_i \cap \boldsymbol{e} = \emptyset$ therefore $\mathrm{Part}_{\boldsymbol{g}_i}(F) = \mathrm{Part}_{\boldsymbol{g}_i}(\mathrm{Cont}_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains:

i. an edge $\in \boldsymbol{g}_j$. As $(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r) \in \mathcal{P}(\mathrm{Cont}_{\boldsymbol{e}}(F))$, $j<i$.

ii. an edge $\in \boldsymbol{f}_j$. It is good because $\boldsymbol{f}_j$ is before $\boldsymbol{g}_i$.

Thus $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q,\boldsymbol{g}_1,\dots,\boldsymbol{g}_r) \in \mathcal{P}(F)$.

So $g\left((\boldsymbol{e},(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q),(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r))\right) \in \mathbb{E}(F)$.

Finally, we easily see that $f \circ g = Id_{\mathbb{F}(F)}$ and $g \circ f = Id_{\mathbb{E}(F)}$. □

We now prove proposition 30. By induction on the edge degree n of $F \in \mathbf{C}_{CK}^{\mathcal{D}}$. If $n=1$, $F = \tdeux{a}$ with $a \in \mathcal{D}$. Then $\Phi(\tdeux{a}) = \varphi(\tdeux{a})$ and formula (16) is true. Suppose that $n \geq 2$ and that the property is true in degrees $k<n$. Then

$$\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= (\Phi \circ \Phi) \circ \tilde{\Delta}_{\mathbf{C}_{CK}^{\mathcal{D}}}(F) \\
&= \sum_{\boldsymbol{e} \Vdash E(F)} \Phi\left(\mathrm{Part}_{\boldsymbol{e}}(F)\right) \otimes \Phi\left(\mathrm{Cont}_{\boldsymbol{e}}(F)\right) \\
&= \sum_{\boldsymbol{e} \Vdash E(F)} \left(\sum_{(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q) \in \mathcal{P}(\mathrm{Part}_{\boldsymbol{e}}(F))} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{f}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{f}_q}}(F)) \right) \\
&\qquad \otimes \left(\sum_{(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r) \in \mathcal{P}(\mathrm{Cont}_{\boldsymbol{e}}(F))} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{g}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{g}_r}}(F)) \right)
\end{aligned}$$

using induction hypothesis in the last equality. So, with lemma 31,

$$\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= \sum_{(\boldsymbol{e},(\boldsymbol{f}_1,\dots,\boldsymbol{f}_q),(\boldsymbol{g}_1,\dots,\boldsymbol{g}_r)) \in \mathbb{F}(F)} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{f}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{f}_q}}(F)) \\
&\qquad \otimes \varphi(\mathrm{Cont}_{\overline{\boldsymbol{g}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{g}_r}}(F)) \\
&= \sum_{((\boldsymbol{e}_1,\dots,\boldsymbol{e}_k),p) \in \mathbb{E}(F)} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad \otimes \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_{p+1}}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_k}}(F)) \\
&= \sum_{(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k) \in \mathcal{P}(F)} \sum_{1 \leq p \leq k-1} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad \otimes \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_{p+1}}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_k}}(F)).
\end{aligned}$$

Therefore

$$\Phi(F) = \sum_{(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k) \in \mathcal{P}(F)} \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1}}(F)) \dots \varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_k}}(F))$$

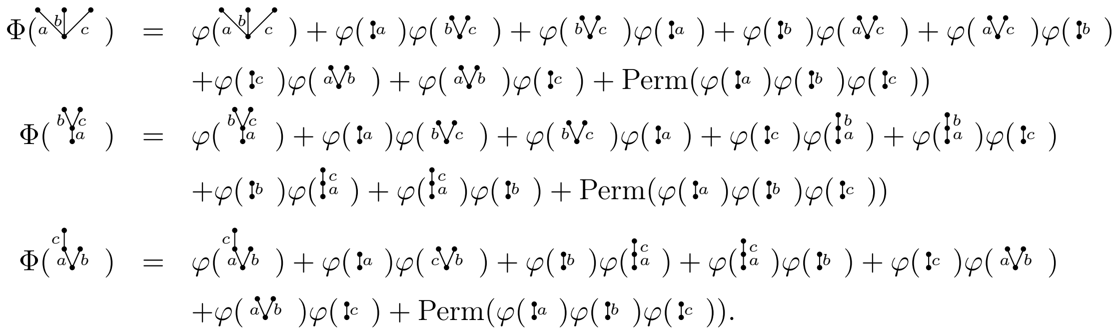
and by induction, we have the result. □

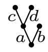
Examples. We introduce a notation. If $w = w_1 \dots w_n$ is a $\mathcal{D}$-word, we denote by $\mathrm{Perm}(w)$ the sum of all $\mathcal{D}$-words whose letters are $w_1,\dots,w_n$. For example, $\mathrm{Perm}(abc) = abc+acb+bac+bca+cab+cba$.

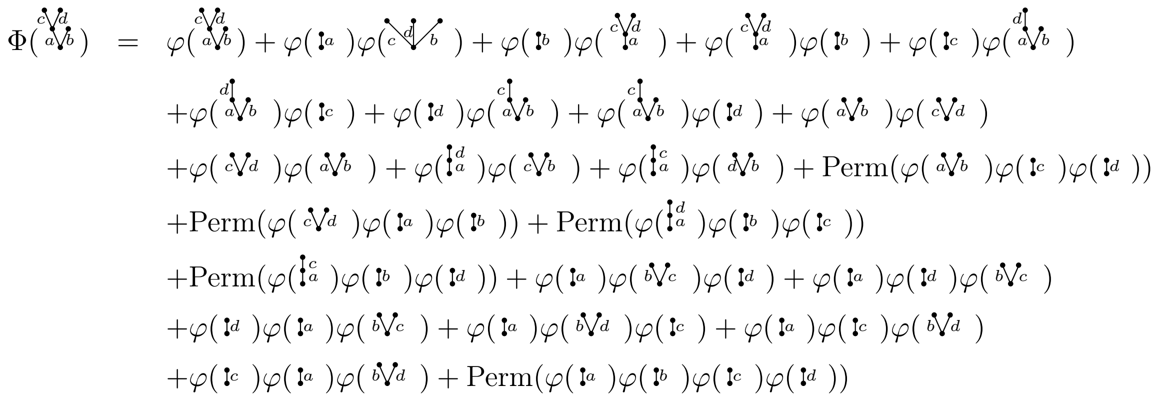
- In edge degree 1, $\Phi(\tdeux{a}) = \varphi(\tdeux{a})$.
- In edge degree 2,

$$\begin{aligned}
\Phi(\ttroisun{a}{b}) &= \varphi(\ttroisun{a}{b}) + \varphi(\tdeux{a})\varphi(\tdeux{b}) + \varphi(\tdeux{b})\varphi(\tdeux{a}) \\
\Phi(\ttroisdeux{a}{b}) &= \varphi(\ttroisdeux{a}{b}) + \varphi(\tdeux{a})\varphi(\tdeux{b}) + \varphi(\tdeux{b})\varphi(\tdeux{a}).
\end{aligned}$$

- In edge degree 3,



- Finally, in edge degree 4, with the tree ,



5.4 From $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 32 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}_{CK}^{\mathcal{D}} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{C}_{CK}^{\mathcal{D}} \arrow[r, "\Phi"] & \mathbf{Csh}^{\mathcal{D}} \arrow[u, "\pi"'] \end{tikzcd} \tag{17}$$

is commutative.

Proof. This is the same proof as for theorem 23. □

We give a combinatorial description of the morphism Φ defined in theorem 32. For this, we give the following definition:

Definition 33 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized and contracted partition of F is a l-uplet $(f_1,\dots,f_l)$ such that:*

1. *for all $1 \le i \le l$, $f_i = (\boldsymbol{e}_1^i,\dots,\boldsymbol{e}_{k_i}^i)$ is a k_i-uplet of subsets of $E(F)$,*
2. *$(\boldsymbol{e}_1^1,\dots,\boldsymbol{e}_{k_1}^1,\boldsymbol{e}_1^2,\dots,\boldsymbol{e}_{k_l}^l) \in \mathcal{P}(F)$,*

3. *if* $\mathrm{Part}_{\boldsymbol{e}_p^i}(F)$ *and* $\mathrm{Part}_{\boldsymbol{e}_q^i}(F)$ *are two disconnected components of* F *and if the shortest path in* F *between* $\mathrm{Part}_{\boldsymbol{e}_p^i}(F)$ *and* $\mathrm{Part}_{\boldsymbol{e}_q^i}(F)$ *contains an edge* $\in \boldsymbol{e}_r^j$, *then* $j > i$.

We shall denote by $\mathcal{P}_c(F)$ *the set of generalized and contracted partitions of* F.

Proposition 34 *Let* F *be a nonempty forest* $\in \mathbf{C}_{CK}^{\mathcal{D}}$. *Then*

$$\begin{aligned}\Phi(F) \quad = \quad \sum_{\substack{(f_1,\dots,f_l)\in\mathcal{P}_c(F)\\ f_i=(\boldsymbol{e}_1^i,\dots,\boldsymbol{e}_{k_i}^i)}} & \left(\left[\varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1^1}}(F))\dots\varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_{k_1}^1}}(F))\right]^{(k_1)}\dots\right.\\ & \left.\dots\left[\varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_1^l}}(F))\dots\varphi(\mathrm{Cont}_{\overline{\boldsymbol{e}_{k_l}^l}}(F))\right]^{(k_l)}\right).\end{aligned} \tag{18}$$

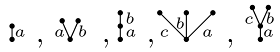
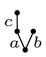
Proof. It suffices to adapt the proof of proposition 30. Note that, if T is a rooted tree and $\boldsymbol{e} \models E(T)$, $\mathrm{Cont}_{\boldsymbol{e}}(T)$ is a tree and $\mathrm{Part}_{\boldsymbol{e}}(T)$ is a forest. So there is possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T)$. Remark that

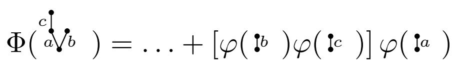
- the trees of $\mathrm{Part}_{\boldsymbol{e}}(T)$ are disconnected components of T and they appear to the left of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T)$,
- the edges of $\overline{\boldsymbol{e}}$ between two disconnected components of $\mathrm{Part}_{\boldsymbol{e}}(T)$ in T are edges of $\mathrm{Cont}_{\boldsymbol{e}}(T)$ and thus they appear to the right of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T)$.

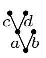
We deduce formula (18). $\square$

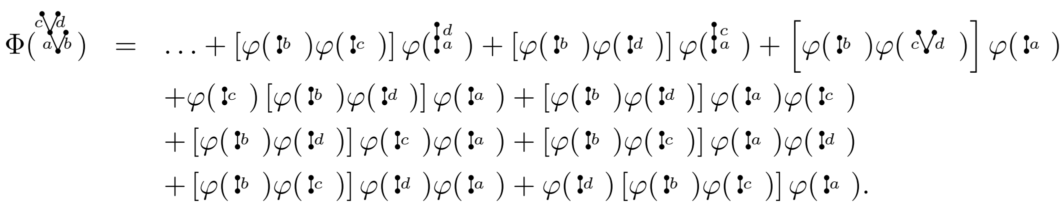
Remark. In the expression of $\Phi(F)$ (formula (18)), we find the terms of (16) and other terms with contractions for the product $[\cdot,\cdot]$. Taking $[\cdot,\cdot]=0$, we obtain (16) again.

Examples. From the examples at the end of section 5.3, we give the other terms with contractions for the product $[\cdot,\cdot]$.

- There are no terms with contractions for the following trees:  .
- For the tree  ,



- For the tree ,



References

[CEFM11] D. Calaque, K. Ebrahimi-Fard, and D. Manchon, *Two interacting Hopf algebras of trees*, Advances in Appl. Math. **47** (2011), 282–308, arXiv:0806.2238.

[CK98] A. Connes and D. Kreimer, *Hopf algebras, Renormalization and Noncommutative geometry*, Comm. Math. Phys. **199** (1998), 203–242, arXiv:hep-th/9808042.

[CK00] ———, *Renormalization in quantum field theory and the Riemann-Hilbert problem I. the Hopf algebra of graphs and the main theorem*, Comm. Math. Phys. **210** (2000), 249–273, arXiv:hep-th/9912092.

[CK01] ———, *Renormalization in quantum field theory and the Riemann-Hilbert problem II. β-function, diffeomorphisms and the renormalization group*, Comm. Math. Phys. **216** (2001), 215–241, arXiv:hep-th/0003188.

[DHT02] G. Duchamp, F. Hivert, and J.-Y. Thibon, *Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras*, Int. J. Algebra Comput. **12** (2002), no. 671, arXiv:math/0105065.

[EV04] J. Ecalle and B. Vallet, *The arborification-coarborification transform: analytic, combinatorial, and algebraic aspects*, Ann. Fac. Sc. Toulouse **13** (2004), no. 4, 575–657.

[Foi02a] L. Foissy, *Les algèbres de Hopf des arbres enracinés décorés, I*, Bull. Sci. Math. **126** (2002), no. 3, 193–239, arXiv:math/0105212.

[Foi02b] ———, *Les algèbres de Hopf des arbres enracinés décorés, II*, Bull. Sci. Math. **126** (2002), no. 4, 249–288, arXiv:math/0105212.

[Foi12] ———, *Ordered forests and parking functions*, Int. Math. Res. Notices **2011** (2012), arXiv:1007.1547.

[Foi13] ———, *The Hopf algebra of Fliess operators and its dual pre-Lie algebra*, preprint (2013), arXiv:1304.1726.

[FU12] L. Foissy and J. Unterberger, *Ordered forests, permutations and iterated integrals*, Int. Math. Res. Notices (2012), arXiv:1004.5208.

[GL90] R.L. Grossman and R.G. Larson, *Hopf-algebraic structure of combinatorial objects and differential operators*, Israel J. Math. **72** (1990), no. 1–2, 109–117, arXiv:0711.3877.

[Hof00] M.E. Hoffman, *Quasi-shuffle products*, Journal of Algebraic Combinatorics **11** (2000), 49–68, arXiv:math/9907173.

[Hol03] R. Holtkamp, *Comparaison of Hopf algebras on trees*, Arch. Math. (Basel) **80** (2003), no. 4, 368–383.

[LR06] J.L. Loday and M. Ronco, *On the structure of cofree hopf algebra*, J. reine angew. Math. **592** (2006), 123–155, arXiv:math/0405330.

[LR10] ———, *Combinatorial hopf algebras*, Amer. Math. Soc., Quanta of maths (2010), no. 11, 347–383, arXiv:0810.0435.

[LV12] J.L. Loday and B. Vallette, *Algebraic Operads*, Springer, 2012.

[Moe01] I. Moerdijk, *On the Connes-Kreimer construction of Hopf algebras*, Contemp. Math. **271** (2001), 311–321, arXiv:math-ph/9907010.

[MR95] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982.

[MS11] D. Manchon and A. Saïdi, *Lois pré-Lie en interaction*, Communications in Algebra **39** (2011), 3662–3680, arXiv:0811.2153.

[NT06] J.-C. Novelli and J.-Y. Thibon, *Polynomial realizations of some trialgebras*, arXiv:math/0605061.

[Reu93] C. Reutenauer, *Free Lie algebras*, The Clarendon Press, 1993.

[Slo] N.J.A. Sloane, *On-line encyclopedia of integer sequences*, http://www.research.att.com/ ˜njas/ sequences/ Seis.html.

[Sta02] R.P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge University Press, 2002.