

Sommes doubles

Exercice 1

$$\begin{aligned}
 \sum_{1 \leq i \leq j \leq n} ij &= \sum_{j=1}^n \left(\sum_{i=1}^j ij \right) = \sum_{j=1}^n \left(j \sum_{i=1}^j i \right) = \sum_{j=1}^n \left(j \times \frac{j(j+1)}{2} \right) = \frac{1}{2} \left(\sum_{j=1}^n j^3 + \sum_{j=1}^n j^2 \right) \\
 &= \frac{1}{2} \left(\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \right) = \frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{24} \\
 &= \frac{n(n+1)(3n(n+1) + 2(2n+1))}{24} = \frac{n(n+1)(3n^2 + 3n + 4n + 2)}{24} \\
 &= \frac{n(n+1)(3n^2 + 7n + 2)}{24}.
 \end{aligned}$$

2.

$$\begin{aligned}
 \sum_{1 \leq i < j \leq n} (i+j) &= \sum_{j=2}^n \left(\sum_{i=1}^{j-1} (i+j) \right) = \sum_{j=2}^n \left(\sum_{i=1}^{j-1} i + \sum_{i=1}^{j-1} j \right) = \sum_{j=2}^n \left(\frac{(j-1)j}{2} + (j-1)j \right) \\
 &= \sum_{j=2}^n \frac{3(j^2-j)}{2} = \frac{3}{2} \sum_{j=2}^n j^2 - \frac{3}{2} \sum_{j=2}^n j = \frac{3}{2} \left(\sum_{j=1}^n j^2 - 1 \right) - \frac{3}{2} \left(\sum_{j=1}^n j - 1 \right) \\
 &= \frac{3}{2} \frac{n(n+1)(2n+1)}{6} - \frac{3}{2} \frac{3n(n+1)}{2} + \frac{3}{2} = \frac{3}{2} \frac{n(n+1)((2n+1)-3)}{6} \\
 &= \frac{n(n+1)(2n-2)}{4} = \frac{(n-1)n(n+1)}{2}.
 \end{aligned}$$

3.

$$\sum_{0 \leq i, j \leq n} 2^{i+j} = \sum_{j=0}^n \left(\sum_{i=0}^n 2^{i+j} \right) = \sum_{j=0}^n \left(2^j \sum_{i=0}^n 2^i \right) = \sum_{j=0}^n 2^j \frac{1-2^{n+1}}{1-2} = \frac{1-2^{n+1}}{-1} \frac{1-2^{n+1}}{-1} = (1-2^{n+1})^2.$$

4.

$$\begin{aligned}
 \sum_{1 \leq i \leq j \leq n} \frac{i}{j} &= \sum_{j=1}^n \left(\sum_{i=1}^j \frac{i}{j} \right) = \sum_{j=1}^n \frac{1}{j} \left(\sum_{i=1}^j i \right) = \sum_{j=1}^n \frac{1}{j} \times \frac{j(j+1)}{2} = \sum_{j=1}^n \frac{j+1}{2} = \frac{1}{2} \sum_{j=1}^n j + \frac{1}{2} \sum_{j=1}^n 1 \\
 &= \frac{1}{2} \times \frac{n(n+1)}{2} + \frac{1}{2} \times n = \frac{n(n+3)}{4}.
 \end{aligned}$$

Exercice 2 1. (a) La valeur de S en sortie correspond à la valeur de la somme $\sum_{1 \leq i, j \leq n} \min(i, j)$.

(b) On décompose S en trois sommes, en distinguant les cas où $i < j$, $i = j$ et $i > j$:

$$\begin{aligned}
 \sum_{1 \leq i, j \leq n} \min(i, j) &= \sum_{1 \leq i < j \leq n} \min(i, j) + \sum_{1 \leq i = j \leq n} \min(i, j) + \sum_{1 \leq j < i \leq n} \min(i, j) \\
 &= \sum_{1 \leq i < j \leq n} i + \sum_{1 \leq i \leq n} i + \sum_{1 \leq j < i \leq n} j.
 \end{aligned}$$

(c)

$$\begin{aligned}
 \sum_{1 \leq i, j \leq n} \min(i, j) &= \sum_{1 \leq i < j \leq n} i + \sum_{1 \leq i \leq n} i + \sum_{1 \leq j < i \leq n} j = 2 \sum_{1 \leq i < j \leq n} i + \sum_{1 \leq i \leq n} i \\
 &= 2 \sum_{j=2}^n \left(\sum_{i=1}^{j-1} i \right) + \sum_{1 \leq i \leq n} i = 2 \sum_{j=2}^n \frac{(j-1)j}{2} + \frac{n(n+1)}{2} \\
 &= \sum_{j=2}^n j^2 - \sum_{j=2}^n j + \frac{n(n+1)}{2} = \sum_{j=1}^n j^2 - 1 - \sum_{j=1}^n j + 1 + \frac{n(n+1)}{2} \\
 &= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.
 \end{aligned}$$

2. Voici la procédure pour calculer T_n :

```

n=input('Donner une valeur de n : ')
T=0
for i=1:n do
    for j=1:n do
        if i>j then
            T=T+i
        else
            T=T+j
        end
    end
end
disp(T)

```

3. (a) Voici la procédure pour calculer U_n :

```

n=input('Donner une valeur de n : ')
U=0
for i=1:n do
    for j=1:n do
        U=U+abs(i-j)
    end
end
disp(U)

```

(b)

$$\begin{aligned}
 \sum_{1 \leq i, j \leq n} |i-j| &= \sum_{1 \leq i < j \leq n} |i-j| + \sum_{1 \leq i \leq n} 0 + \sum_{1 \leq j < i \leq n} |i-j| = \sum_{1 \leq i < j \leq n} (j-i) + \sum_{1 \leq j < i \leq n} (i-j) \\
 &= 2 \sum_{1 \leq i < j \leq n} (j-i) = 2 \sum_{j=2}^n \left(\sum_{i=1}^j (j-i) \right) = 2 \sum_{j=2}^n \left(j^2 - \frac{j(j+1)}{2} \right) = 2 \sum_{j=2}^n \frac{j^2-j}{2} \\
 &= \sum_{j=2}^n (j^2-j) = \sum_{j=1}^n (j^2-j) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)((2n+1)-3)}{6} \\
 &= \frac{(n-1)n(n+1)}{3}.
 \end{aligned}$$

Exercice 3 1.

$$\sum_{1 \leq i \leq j \leq n} q^j = \sum_{j=1}^n \left(\sum_{i=1}^j q^j \right) = \sum_{j=1}^n j q^j.$$

2. Avec la formule d'interversion des indices, on a :

$$\begin{aligned}
 \sum_{j=1}^n j q^j &= \sum_{1 \leq i \leq j \leq n} q^j = \sum_{i=1}^n \left(\sum_{j=i}^n q^j \right) = \sum_{i=1}^n \left(\sum_{k=0}^{n-i} q^{k+i} \right) = \sum_{i=1}^n \left(q^i \sum_{k=0}^{n-i} q^k \right) = \sum_{i=1}^n \left(q^i \frac{1-q^{n-i+1}}{1-q} \right) \\
 &= \sum_{i=1}^n \frac{q^i - q^{n+1}}{1-q} = \frac{1}{1-q} \left(\sum_{i=1}^n q^i - \sum_{i=1}^n q^{n+1} \right) = \frac{1}{1-q} \left(\sum_{k=0}^{n-1} q^{k+1} - \sum_{i=1}^n q^{n+1} \right) \\
 &= \frac{1}{1-q} \left(q \sum_{k=0}^{n-1} q^k - \sum_{i=1}^n q^{n+1} \right) = \frac{1}{1-q} \left(q \frac{1-q^n}{1-q} - n q^{n+1} \right) \\
 &= \frac{q - q^{n+1} - n q^{n+1} (1-q)}{(1-q)^2} = \frac{q - (n+1) q^{n+1} + n q^{n+2}}{(1-q)^2}.
 \end{aligned}$$