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Structures Hopf-algébriques et opéradiques sur différentes familles d'arbres

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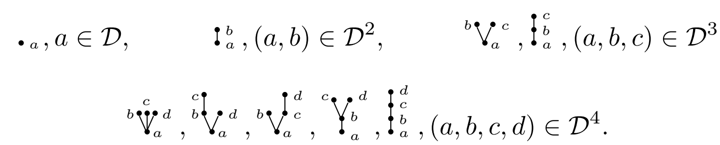
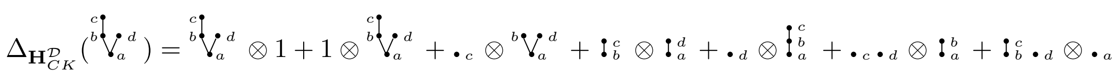
Introduction

L'algèbre de Hopf de Connes-Kreimer des forêts enracinées (dont les sommets sont éventuellement décorés) $\mathbf{H}_{CK}^{\mathcal{D}}$ est introduite dans [CK98, Moe01]. Cette algèbre de Hopf graduée (par le nombre de sommets), commutative et non cocommutative est utilisée dans [CK00, CK01] pour étudier un problème de renormalisation dans la théorie des champs quantiques. Voici par exemple les forêts enracinées décorées par un ensemble $\mathcal{D}$ de degré inférieur à 4 :

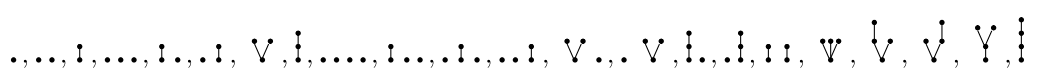


Le coproduit est défini à partir de coupes admissibles sur les arêtes. Par exemple,

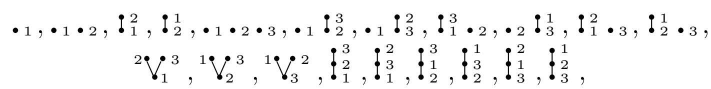


Une version non commutative, l'algèbre de Hopf $\mathbf{H}_{NCK}^{\mathcal{D}}$ des forêts enracinées planes, est donnée dans [Foi02a, Foi02b, Hol03]. $\mathbf{H}_{CK}^{\mathcal{D}}$ et $\mathbf{H}_{NCK}^{\mathcal{D}}$ vérifie chacune une propriété universelle en cohomologie de Hochschild. Avec un ordre total sur les sommets, on défini l'algèbre de Hopf des forêts ordonnées $\mathbf{H}_o$ et, avec une condition de croissance en plus, on obtient l'algèbre de Hopf des forêts ordonnées en tas $\mathbf{H}_{ho}$ (voir [FU10, GL90]). Par exemple,

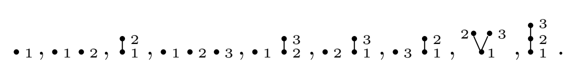
– les forêts enracinées planes de degrés ≤ 4 :



– les forêts ordonnées de degrés ≤ 3 :

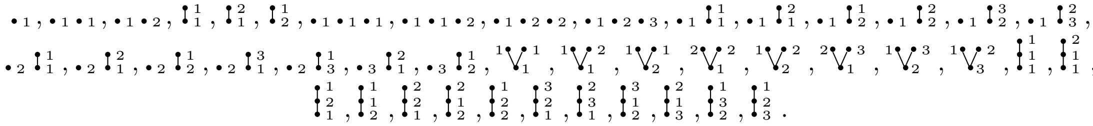


– les forêts ordonnées en tas de degrés ≤ 3 :

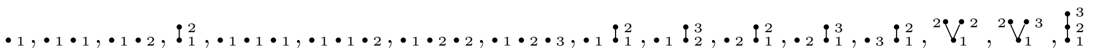


Par ailleurs, l'algèbre de Hopf $\mathbf{FQSym}^{\mathcal{D}}$ des fonctions quasi-symétriques (éventuellement décorées) est introduite dans [DHT02, MR95]. Dans [FU10], L. Foissy et J. Unterberger prouvent qu'il existe un morphisme d'algèbres de Hopf, décrit au théorème 6, de $\mathbf{H}_o$ dans $\mathbf{FQSym}$. Il montre de plus que sa restriction à $\mathbf{H}_{ho}$ est un isomorphisme d'algèbres de Hopf. A partir de $\mathbf{FQSym}^{\mathcal{D}}$, on peut construire l'algèbre de Hopf $\mathbf{Sh}^{D}$ des mots en l'alphabet $\mathcal{D}$ dont le produit est le battage des mots et le coproduit est la déconcaténation (voir [Hof00]).

Nous définissons ici une généralisation des forêts ordonnées : les forêts préordonnées. Un préordre est une relation binaire réflexive et transitive. Une forêt préordonnée est une forêt enracinée avec un préordre total sur ses sommets. Voici les forêts préordonnées de degrés ≤ 3 :



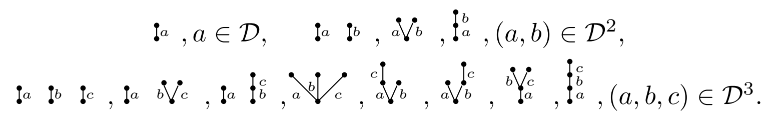
Nous prouvons que l'algèbre des forêts préordonnées $\mathbf{H}_{po}$ est une algèbre de Hopf pour le coproduit de coupes. Avec une condition de croissance, nous définissons aussi l'algèbre des forêts préordonnées en tas $\mathbf{H}_{hpo}$. Par exemple, en degré ≤ 3 :

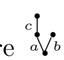


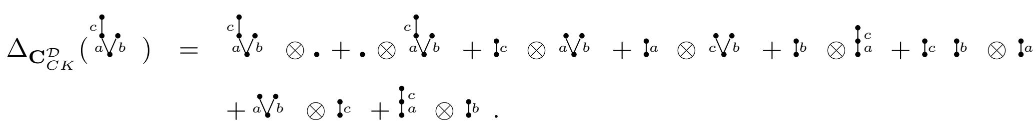
Nous prouvons que l'algèbre des forêts préordonnées en tas $\mathbf{H}_{hpo}$ est une sous-algèbre de Hopf de $\mathbf{H}_{po}$.

L'algèbre de Hopf des mots tassés $\mathbf{WQSym}^*$ est introduite par J.-C. Novelli et J.-Y. Thibon dans [NT06]. A partir d'une version décorée du dual gradué $\mathbf{WQSym}^{\mathcal{D}}$ de $\mathbf{WQSym}$, on définit l'algèbre de Hopf $\mathbf{Csh}^{\mathcal{D}}$ des mots en l'alphabet $\mathcal{D}$ dont le produit est le battage contractant des mots et le coproduit est la déconcaténation (voir [Hof00]). En remplaçant dans [FU10] les forêts ordonnées par les forêts préordonnées et les fonctions quasi-symétriques par les mots tassés, nous établissons un lien entre $\mathbf{H}_{po}$ et $\mathbf{WQSym}^*$: nous démontrons au théorème 11 qu'il existe un morphisme d'algèbres de Hopf de $\mathbf{H}_{po}$ dans $\mathbf{WQSym}^*$. On montre de plus que la restriction de ce morphisme à $\mathbf{H}_{hpo}$ est injective.

Dans [CEFM11], D. Calaque, K. Ebrahimi-Fard et D. Manchon introduisent, dans le cas commutatif, un autre coproduit sur les forêts enracinées donné en contractant certaines arêtes (voir aussi [MS11]). Ce coproduit de contraction est défini sur un quotient $\mathbf{C}_{CK}$ de $\mathbf{H}_{CK}$ et $\mathbf{C}_{CK}$ est alors une algèbre de Hopf graduée par le nombre d'arêtes, commutative et non cocommutative. Nous donnons une version décorée $\mathbf{C}_{CK}^{\mathcal{D}}$ de $\mathbf{C}_{CK}$, où ce sont les arêtes qui sont décorées. Voici les forêts appartenant à $\mathbf{C}_{CK}^{\mathcal{D}}$ de degrés ≤ 3 :



En appliquant le coproduit de contraction à l'arbre  , on obtient :



Nous définissons deux opérations $\curlyvee$ et $\triangleright$ sur l'espace vectoriel $\mathbf{T}_{CK}^{\mathcal{D}}$ de base les arbres de $\mathbf{C}_{CK}^{\mathcal{D}}$. On démontre alors que $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ est une algèbre commutative pré-Lie, c'est-à-dire que $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee)$ est une algèbre commutative, $(\mathbf{T}_{CK}^{\mathcal{D}}, \triangleright)$ est une algèbre pré-Lie et que, pour tout $x, y, z \in \mathbf{T}_{CK}^{\mathcal{D}}$,

$$x \triangleright (y \curlyvee z) = (x \triangleright y) \curlyvee z + (x \triangleright z) \curlyvee y.$$

On prouve que $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ est engendrée comme algèbre commutative pré-Lie par les arbres  , $d \in \mathcal{D}$.

Nous construisons une version non commutative de $\mathbf{C}_{CK}$. Pour cela, nous construisons des quotients $\mathbf{C}_{NCK}$, $\mathbf{C}_{ho}$, $\mathbf{C}_{o}$, $\mathbf{C}_{hpo}$ et $\mathbf{C}_{po}$ de $\mathbf{H}_{NCK}$, $\mathbf{H}_{ho}$, $\mathbf{H}_{o}$, $\mathbf{H}_{hpo}$ et $\mathbf{H}_{po}$. Nous définissons un coproduit de contraction sur ces quotients. Nous prouvons aux théorèmes 21 et 22 que $\mathbf{C}_{ho}$, $\mathbf{C}_{o}$, $\mathbf{C}_{hpo}$ et $\mathbf{C}_{po}$ sont des algèbres de Hopf et que $\mathbf{C}_{NCK}$ est un comodule à gauche sur l'algèbre de Hopf $\mathbf{C}_{ho}$.

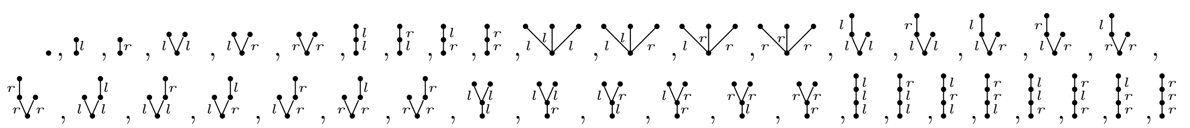
Nous étudions ensuite les morphismes d'algèbres de Hopf de $\mathbf{H}_{CK}^{\mathcal{D}}$ ou $\mathbf{C}_{CK}^{\mathcal{D}}$ dans $\mathbf{Sh}^{\mathcal{D}}$ ou $\mathbf{Csh}^{\mathcal{D}}$. Nous donnons dans chacun des cas une description combinatoire de ces morphismes utilisant les objets déjà introduits. On remarque en particulier que le coproduit de contraction et les forêts préordonnées apparaissent naturellement. Par ailleurs, dans le cas des morphismes d'algèbres de Hopf de $\mathbf{H}_{CK}^{\mathcal{D}}$ dans $\mathbf{Sh}^{\mathcal{D}}$, on obtient une extension du morphisme d'arborification décrit par J. Ecalle et B. Vallet dans [EV04].

Dans [Foi10], L. Foissy introduit la notion d'algèbre de greffe à droite (ou $\mathcal{RG}$-algèbre) et d'algèbre de greffe à gauche (ou $\mathcal{LG}$-algèbre). Une $\mathcal{RG}$-algèbre est une algèbre associative $(A, *)$ munie d'une greffe à droite $\prec$ telle que, pour tout $x, y, z \in A$,

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z), \\ (x * y) \prec z &= x * (y \prec z). \end{aligned}$$

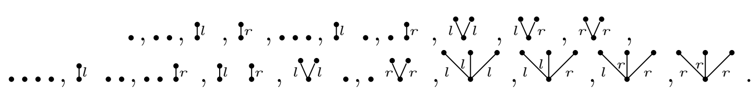
Il prouve que l'algèbre de forêts enracinées planes est l'algèbre de greffe à droite libre à un générateur et démontre que $(\mathcal{A}ss, \mathcal{RG}, \mathcal{M}ag)$ est un bon triplet d'opérades (voir [Lod08] pour la notion de triplet d'opérades). D'autre part, P. Leroux définit dans [Ler03, Ler08] les $\mathcal{L}$-algèbres, qui sont des algèbres munies de deux opérations $\succ$ et $\prec$ avec la relation d'entrelacement $x \succ (y \prec z) = (x \succ y) \prec z$.

Nous introduisons ici la notion d'algèbres bigreffe. Une algèbre bigreffe $(A, *, \succ, \prec)$ (ou $\mathcal{BG}$-algèbre) est une algèbre de greffe à gauche $(A, *, \succ)$, une algèbre de greffe à droite $(A, *, \prec)$ et une $\mathcal{L}$-algèbre $(A, \succ, \prec)$. On construit l'algèbre bigreffe libre à un générateur. Pour cela, on considère l'algèbre $\mathbf{H}_{BG}$ des arbres enracinés plans dont les arêtes sont décorées avec deux décorations l et r (avec une condition d'ordre sur les décorations). Voici par exemple les arbres de $\mathbf{H}_{BG}$ de degrés en sommets ≤ 4 :



Nous définissons sur $\mathbf{H}_{BG}$ un opérateur de greffe noté B_{BG} et nous démontrons que $(\mathbf{H}_{BG}, B_{BG})$ est un objet initial dans une certaine catégorie. Ceci nous permet de munir $\mathbf{H}_{BG}$ d'un coproduit, qui peut être décrit à partir de coupes admissibles sur les arêtes, et d'un couplage de Hopf. On prouve au théorème 46 que l'idéal d'augmentation $\mathbf{M}_{BG}$ de $\mathbf{H}_{BG}$ est la $\mathcal{BG}$-algèbre libre à un générateur. On peut alors calculer la dimension de l'opérade bigreffe $\mathcal{BG}$ et donner une description combinatoire de la composition.

L'opérade bigreffe est une opérade binaire et quadratique. Elle admet donc une opérade duale et nous donnons une présentation de son dual de Koszul $\mathcal{BG}^!$. Nous décrivons la $\mathcal{BG}^!$-algèbre libre à un générateur. C'est un quotient de l'algèbre bigreffe $\mathbf{H}_{BG}$ dont les forêts de degrés ≤ 4 sont :



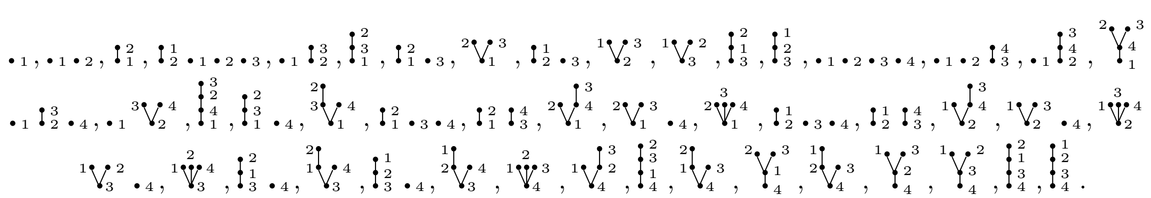
Nous utilisons celle-ci pour construire l'homologie associée à une $\mathcal{BG}$-algèbre. En utilisant une méthode de réécriture (voir [DK10, Hof10, LV12]), on prouve que l'opérade bigreffe est Koszul et on donne des bases de PBW de $\mathcal{BG}$ et de $\mathcal{BG}^!$.

Nous nous intéressons ensuite aux relations de compatibilités entre $*, \succ, \prec$ et le coproduit de déconcaténation $\tilde{\Delta}_{\mathcal{A}ss}$. Ceci nous permet d'introduire la notion de bialgèbres bigreffes infinitésimales : c'est une famille $(A, *, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ où $*, \succ, \prec$: $A \otimes A \to A$, $\tilde{\Delta}_{\mathcal{A}ss} : A \to A \otimes A$, telle que $(A, *, \succ, \prec)$ est une algèbre bigreffe et pour tout $a, y \in A$:

$$\left\{ \begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x * y) & = & (x \otimes 1) * \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x) * (1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array} \right.$$

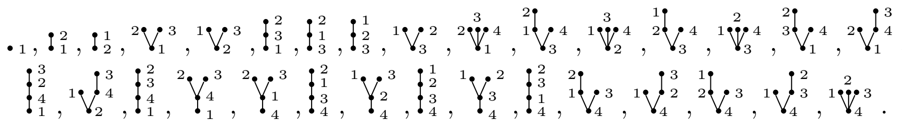
Nous démontrons au théorème 85 un analogue des théorèmes de Poincaré-Birkhoff-Witt et de Cartier-Milnor-Moore dans le cas des bialgèbres bigreffes infinitésimales (en utilisant les résultats de [Lod08]). Pour cela, nous montrons que le sous-espace des éléments primitifs d'une bialgèbre bigreffe infinitésimale est une $\mathcal{L}$-algèbre ce qui nous permet d'obtenir une description combinatoire de la $\mathcal{L}$-algèbre libre à un générateur. On définit l'algèbre bigreffe enveloppante universelle d'une $\mathcal{L}$-algèbre. On prouve alors le théorème de rigidité annoncé et on en déduit que $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ est un bon triplet d'opérades.

Dans [Men02], F. Menous étudie certains ensembles de probabilités associés à une variable aléatoire sur $\mathbb{R}$. Pour cela, il est amené à construire par récurrence, avec un formalisme proche de celui du calcul moulien, un ensemble de forêts ordonnées noté $\mathbb{G}$: par exemple, en degré en sommets ≤ 4, les forêts de $\mathbb{G}$ sont



Nous proposons d'étudier cet ensemble $\mathbb{G}$. Pour cela, nous introduisons deux opérateurs de greffes B^+ et B^-, à partir desquels nous proposons une construction de l'ensemble $\mathbb{G}$. Nous donnons différentes propriétés combinatoires sur $\mathbb{G}$. Ceci nous permet en particulier de démontrer que l'algèbre $\mathbf{B}^\infty = \mathbb{K}[\mathbb{G}]$ est une algèbre de Hopf (proposition 93) et de calculer sa série formelle. Toujours avec les opérateurs B^+ et B^-, nous construisons une suite croissante $\left(\mathbb{G}^i\right)_{i\geq 1}$ de sous-ensemble de $\mathbb{G}$. Nous prouvons à la proposition 96 que pour tout entier $i \geq 1$, $\mathbf{B}^i = \mathbb{K}[\mathbb{G}^i]$ est une algèbre de Hopf et on obtient ainsi un "dévissage" de l'algèbre de Hopf $\mathbf{B}^\infty$ avec les inclusions $\mathbf{B}^1 \subseteq \ldots \subseteq \mathbf{B}^i \subseteq \mathbf{B}^{i+1} \subseteq \ldots \subseteq \mathbf{B}^\infty$.

Nous construisons ensuite à l'aide des opérateurs B^+ et B^- un ensemble d'arbres ordonnés $\mathbb{T}$ contenant les arbres de $\mathbb{G}$. Voici les arbres de $\mathbb{T}$ de degrés en sommets ≤ 4 :



On démontre à la proposition 99 que $\mathbf{B} = \mathbb{K}[\mathbb{T}]$ est une algèbre de Hopf et on donne sa série formelle. On munit $\mathbf{B}$ d'une structure de bialgèbre dupliciale dendriforme (voir [Foi07, Foi12, Lod08]). En utilisant les résultats de [Foi12], nous en déduisons que $\mathbf{B}$ est colibre et auto-duale. D'autre part, on définit sur l'algèbre $\mathbf{B}$ deux greffes à gauche et à droite de sorte que $\mathbf{B}$ est une algèbre bigreffe. On démontre enfin au théorème 115 qu'elle est engendrée comme algèbre bigreffe par l'unique arbre de degré 1.

La rédaction de la thèse suit l'ordre des résultats décrits ci-dessus. Le chapitre 1 est consacré à des rappels sur les algèbres de Hopf de forêts (enracinées, enracinées planes, ordonnées et ordonnées en tas). On rappelle aussi la construction de $\mathbf{FQSym}^{\mathcal{D}}$ et de $\mathbf{Sh}^{\mathcal{D}}$.

Dans le chapitre 2, nous introduisons les algèbres de Hopf des forêts préordonnées $\mathbf{H}_{po}$ et préordonnées en tas $\mathbf{H}_{hpo}$. Nous rappelons la construction de $\mathbf{WQSym}^*$ et nous construisons un morphisme d'algèbres de Hopf de $\mathbf{H}_{po}$ dans $\mathbf{WQSym}^*$. Par ailleurs, on définit le coproduit de contraction dans le cas commutatif et non commutatif. On décrit enfin les morphismes d'algèbres de Hopf de $\mathbf{H}_{CK}$ ou $\mathbf{C}_{CK}$ dans $\mathbf{Sh}^{\mathcal{D}}$ ou $\mathbf{Csh}^{\mathcal{D}}$.

Nous nous intéressons dans le chapitre 3 aux algèbres bigreffes. Nous décrivons l'algèbre bigreffe libre à un générateur $\mathbf{H}_{BG}$. On calcul l'opérade duale de l'opérade bigreffe et on prouve de plus qu'elle est Koszul. On prouve enfin que $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ est un bon triplet d'opérades.

Dans le dernier chapitre, nous étudions des algèbres de Hopf de forêts ordonnées $\mathbf{B}^i$, $\mathbf{B}^\infty$ et $\mathbf{B}$ construites à partir de deux opérateurs de greffes B^+ et B^-. Nous prouvons que $\mathbf{B}$ est une algèbre de Hopf colibre et auto-duale. Enfin, on muni $\mathbf{B}$ d'une structure d'algèbre bigreffe et on prouve qu'elle est engendrée par un seul générateur comme algèbre bigreffe.

Notations

Tout les espaces vectoriels, les algèbres et les cogèbres sont définis sur un corps $\mathbb{K}$ de caractéristique nulle. Étant donné un ensemble X, nous noterons par $\mathbb{K}(X)$ l'espace vectoriel engendré par X. Pour tout espaces vectoriels V et W, $V \otimes W$ désignera le produit tensoriel de V par W sur $\mathbb{K}$.

Soit n un entier naturel. On note Σ_n le groupe symétrique d'ordre n ($\Sigma_0 = \{1\}$) et Σ l'union disjointe des Σ_n pour tout $n \geq 0$.

Soit $V = \bigoplus_{n=0}^{\infty} V_n$ un $\mathbb{K}$-espace vectoriel gradué. Nous notons $V^{\circledast} = \bigoplus_{n=0}^{\infty} V_n^*$ le dual gradué de V. Si H est une algèbre de Hopf graduée, $H^{\circledast}$ est aussi une algèbre de Hopf.

Soient V et W deux $\mathbb{K}$-espaces vectoriels. On note $\tau : V \otimes W \to W \otimes V$ l'unique application $\mathbb{K}$-linéaire, appelée *volte*, telle que $\tau(v \otimes w) = w \otimes v$ pour tout $v \otimes w \in V \otimes W$.

Soit (A, Δ, ε) une cogèbre counitaire. Alors :

1. $Ker(\varepsilon)$ est l'idéal d'augmentation de A et on le notera $(A)_+$.
2. Soit $1 \in A$, non nul, tel que $\Delta(1) = 1 \otimes 1$. On note alors $\tilde{\Delta}$ le coproduit non counitaire défini par :

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} Ker(\varepsilon) & \to & Ker(\varepsilon) \otimes Ker(\varepsilon), \\ a & \mapsto & \Delta(a) - a \otimes 1 - 1 \otimes a. \end{array} \right.$$

Chapitre 1

Hopf algebras of trees

Introduction

Ce chapitre est consacré à des rappels sur des algèbres de Hopf de forêts. Nous introduisons l'algèbre de Hopf de Connes-Kreimer des forêts enracinées $\mathbf{H}_{CK}^{\mathcal{D}}$ avec éventuellement des décorations sur les sommets. Nous rappelons la définition du coproduit donné par des coupes admissibles sur les arrêtes (voir [CK98, Moe01] pour plus de détails). Nous donnons une version non commutative de l'algèbre de Connes-Kreimer construite à partir de forêts enracinées planes et notée $\mathbf{H}_{NCK}^{\mathcal{D}}$ (voir [Foi02a, Foi02b, Hol03]). On rappelle que $\mathbf{H}_{CK}^{\mathcal{D}}$ et $\mathbf{H}_{NCK}^{\mathcal{D}}$ vérifient chacune une propriété universelle en cohomologie de Hochschild.

Avec un ordre total sur les sommets d'une forêt, nous définissons l'algèbre de Hopf des forêts ordonnées $\mathbf{H}_o$. En ajoutant en plus une condition de croissance, on obtient l'algèbre de Hopf des forêts ordonnées en tas $\mathbf{H}_{ho}$ (voir [FU10, GL90]). Par ailleurs, nous rappelons la construction de l'algèbre de Hopf $\mathbf{FQSym}^{\mathcal{D}}$ des fonctions quasi-symétriques éventuellement décorées (plus de détails dans [DHT02, MR95]). On donne alors une description du morphisme d'algèbres de Hopf de $\mathbf{H}_o$ dans $\mathbf{FQSym}$ construit par L. Foissy et J. Unterberger dans [FU10]. Enfin, à partir de $\mathbf{FQSym}^{\mathcal{D}}$, nous donnons une construction de l'algèbre de Hopf $\mathbf{Sh}^{\mathcal{D}}$ des mots en l'alphabet $\mathcal{D}$ dont le produit est le produit de battage des mots et le coproduit est la déconcaténation (voir [Hof00]).

Le chapitre est organisé comme suit : nous introduisons dans la première partie l'algèbre de Hopf de Connes-Kreimer $\mathbf{H}_{CK}^{\mathcal{D}}$. Dans une seconde partie, nous décrivons l'algèbre de Hopf $\mathbf{H}_{NCK}^{\mathcal{D}}$ des forêts enracinées planes. La dernière partie est consacrée aux algèbres de Hopf $\mathbf{H}_o$ et $\mathbf{H}_{ho}$ des forêts ordonnées et ordonnées en tas. Nous donnons leurs constructions. Nous définissons $\mathbf{FQSym}^{\mathcal{D}}$ puis $\mathbf{Sh}^{\mathcal{D}}$. Enfin nous décrivons un morphisme d'algèbre de Hopf entre $\mathbf{H}_o$ et $\mathbf{FQSym}$.

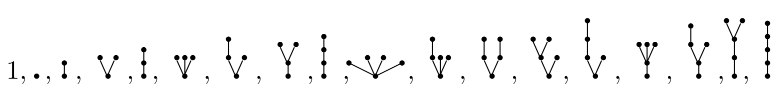
1.1 The Connes-Kreimer Hopf algebra of rooted trees

We briefly recall the construction of the Connes-Kreimer Hopf algebra of rooted trees [CK98]. A *rooted tree* is a finite graph, connected, without loops, with a distinguished vertex called the *root* [Sta02]. We denote by 1 the empty rooted tree. If T is a rooted tree, we denote by R_T the root of T. A *rooted forest* is a finite graph F such that any connected component of F is a rooted tree. The *length* of a forest F, denoted $l(F)$, is the number of connected components of F. The set of vertices of the rooted forest F is denoted by $V(F)$. The *vertices degree* of a forest F, denoted $|F|_v$, is the number of its vertices. The set of edges of the rooted forest F is denoted by $E(F)$. The *edges degree* of a forest F, denoted $|F|_e$, is the number of its edges.

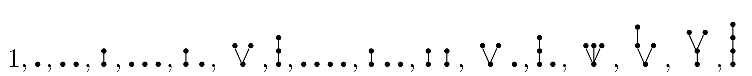
Remark. Let F be a rooted forest. Then $|F|_v = |F|_e + l(F)$.

Examples.

1. Rooted trees of vertices degree ≤ 5 :

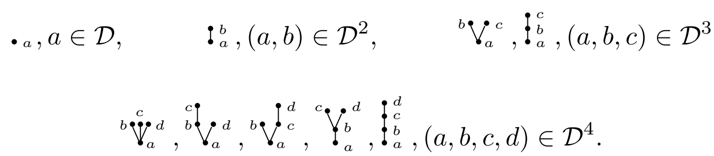


2. Rooted forests of vertices degree ≤ 4 :



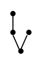
Let $\mathcal{D}$ be a nonempty set. A rooted forest with its vertices decorated by $\mathcal{D}$ is a couple (F,d) with F a rooted forest and $d : V(F) \to \mathcal{D}$ a map.

Examples. Rooted trees with their vertices decorated by $\mathcal{D}$ of vertices degree smaller than 4 :



Let $\mathbb{F}_{\mathbf{H}_{CK}}$ be the set of rooted forests and $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$ the set of rooted forests with their vertices decorated by $\mathcal{D}$. We will denote by $\mathbf{H}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{CK}}$ and by $\mathbf{H}^{\mathcal{D}}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. The set of nonempty rooted trees will be denoted $\mathbb{T}_{\mathbf{H}_{CK}}$ and the set of nonempty rooted trees with their vertices decorated by $\mathcal{D}$ will be denoted $\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. $\mathbf{H}_{CK}$ and $\mathbf{H}^{\mathcal{D}}_{CK}$ are algebras : the product is given by the concatenation of rooted forests.

Let F be a nonempty rooted forest. A *subtree* T of F is a nonempty connected subgraph of F. A *subforest* $T_1 \dots T_k$ of F is a product of disjoint subtrees $T_1, \dots, T_k$ of F. We can give the same definition in the decorated case.

Examples. Consider the tree $T =$  . Then :

- The subtrees of T are $\bullet$ (which appears 4 times), (which appears 3 times), , and (which appear once).
- The subforests of T are $\bullet\bullet$, (which appear 6 times), $\bullet$, $\bullet\bullet\bullet$ (which appear 4 times), , (which appear 3 times) and , , $\bullet\bullet\bullet\bullet$, , , (which appear once).

Let F be a rooted forest. The edges of F are oriented downwards (from the leaves to the roots). If $v, w \in V(F)$, we shall note $v \to w$ if there is an edge in F from v to w and $v \twoheadrightarrow w$ if there is an oriented path from v to w in F. By convention, $v \twoheadrightarrow v$ for any $v \in V(F)$.

If F is a rooted forest and if $v \in V(F)$, then :
- $h(v)$ is the *height* of v, that is to say the number of edges on the oriented path from v to the root of T.
- $f(v)$ is the *fertility* of v, that is to say the cardinality of $\{w \in V(F) \mid w \to v\}$.

The *height* of a rooted forest F is $h(F) = \max(\{h(v),\ v \in V(F)\})$. We shall say that a tree T is a corolla if $h(T) \leq 1$ and a ladder if, for all $v \in V(T)$ that is not a leaf, $f(v) = 1$.

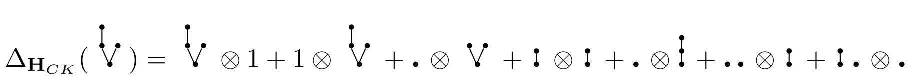
Let $\boldsymbol{v}$ be a subset of $V(F)$. We shall say that $\boldsymbol{v}$ is an admissible cut of F, and we shall write $\boldsymbol{v} \models V(F)$, if $\boldsymbol{v}$ is totally disconnected, that is to say that $v \not\twoheadrightarrow w$ for any couple (v,w) of two different elements of $\boldsymbol{v}$. If $\boldsymbol{v} \models V(F)$, we denote by $Lea_{\boldsymbol{v}}(F)$ the rooted subforest of F obtained by keeping only the vertices above $\boldsymbol{v}$, that is to say $\{w \in V(F),\ \exists v \in \boldsymbol{v},\ w \twoheadrightarrow v\}$, and the edges between these vertices. Note that $\boldsymbol{v} \subseteq Lea_{\boldsymbol{v}}(F)$. We denote by $Roo_{\boldsymbol{v}}(F)$ the rooted subforest obtained by keeping the other vertices and the edges between these vertices.

In particular, if $\boldsymbol{v} = \emptyset$, then $Lea_{\boldsymbol{v}}(F) = 1$ and $Roo_{\boldsymbol{v}}(F) = F$: this is the *empty cut* of F. If $\boldsymbol{v}$ contains all the roots of F, then it contains only the roots of F, $Lea_{\boldsymbol{v}}(F) = F$ and $Roo_{\boldsymbol{v}}(F) = 1$: this is the *total cut* of F. We shall write $\boldsymbol{v} \mid\models V(F)$ if $\boldsymbol{v}$ is a nontotal, nonempty admissible cut of F.

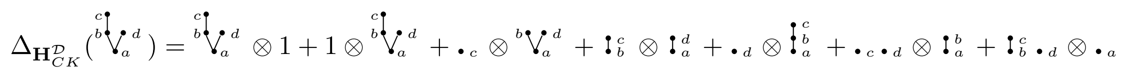
Connes and Kreimer proved in [CK98] that $\mathbf{H}_{CK}$ is a Hopf algebra. The coproduct is the cut coproduct defined for any rooted forest F by :

$$\Delta_{\mathbf{H}_{CK}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \mid\models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example :



In the same way, we can define a cut coproduct on $\mathbf{H}^{\mathcal{D}}_{CK}$. With this coproduct, $\mathbf{H}^{\mathcal{D}}_{CK}$ is also a Hopf algebra. For example :



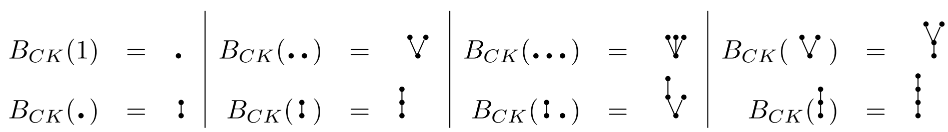
$\mathbf{H}_{CK}$ is graded by the number of vertices. We give some values of the number $f_n^{\mathbf{H}_{CK}}$ of rooted forests of vertices degree n :

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842

This is the sequence A000081 in [Slo].

We define the operator $B_{CK} : \mathbf{H}_{CK} \to \mathbf{H}_{CK}$, which associates, to a forest $F \in \mathbf{H}_{CK}$, the tree obtained by grafting the roots of the trees of F on a common root.

Examples.



It is show in [Moe01] that $(\mathbf{H}_{CK}, B_{CK})$ is an initial objet in the category of couples (A, L), where A is a commutative algebra and $L : A \to A$ any linear operator. Explicitely, one has :

Theorem 1 *Let A be a commutative algebra and let $L : A \to A$ be a linear map. Then there exists a unique algebra morphism $\phi : \mathbf{H}_{CK} \to A$, such that $\phi \circ B_{CK} = L \circ \phi$.*

In particular, $\Delta_{\mathbf{H}_{CK}} : \mathbf{H}_{CK} \to \mathbf{H}_{CK} \otimes \mathbf{H}_{CK}$ is the unique algebra morphism such that

$$\Delta_{\mathbf{H}_{CK}} \circ B_{CK} = B_{CK} \otimes 1 + (Id \otimes B_{CK}) \circ \Delta_{\mathbf{H}_{CK}}.$$

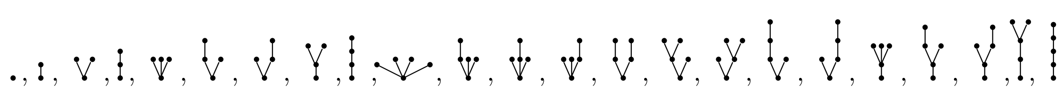
1.2 Hopf algebras of planar trees

We now recall the construction of the noncommutative generalization of the Connes-Kreimer Hopf algebra [Foi02a, Hol03].

A *planar forest* is a rooted forest F such that the set of the roots of F is totally ordered and, for any vertex $v \in V(F)$, the set $\{w \in V(F) \mid w \to v\}$ is totally ordered. Planar forests are represented such that the total orders on the set of roots and the sets $\{w \in V(F) \mid w \to v\}$ for any $v \in V(F)$ is given from left to right. We denote by $\mathbb{T}_{\mathbf{H}_{NCK}}$ the set of nonempty planar trees and $\mathbb{F}_{\mathbf{H}_{NCK}}$ the set of planar forests.

Examples.

1. Planar rooted trees of vertices degree ≤ 5 :

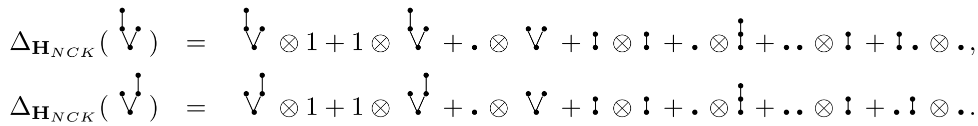


2. Planar rooted forests of vertices degree ≤ 4 :

If $\boldsymbol{v} \models V(F)$, then $Lea_{\boldsymbol{v}}(F)$ and $Roo_{\boldsymbol{v}}(F)$ are naturally planar forests. It is proved in [Foi02a] that the space $\mathbf{H}_{NCK}$ generated by planar forests is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any rooted forest F by :

$$\Delta_{\mathbf{H}_{NCK}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \| \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example :



As in the nonplanar case, there is a decorated version $\mathbf{H}^{\mathcal{D}}_{NCK}$ of $\mathbf{H}_{NCK}$. Moreover, $\mathbf{H}_{NCK}$ is a Hopf algebra graded by the number of vertices. The number $f_n^{\mathbf{H}_{NCK}}$ of planar forests of vertices degree n (equal to the number of planar trees of vertices degree $n+1$) is the n-Catalan number $\frac{(2n)!}{n!(n+1)!}$, see sequence A000108 of [Slo]. We have :

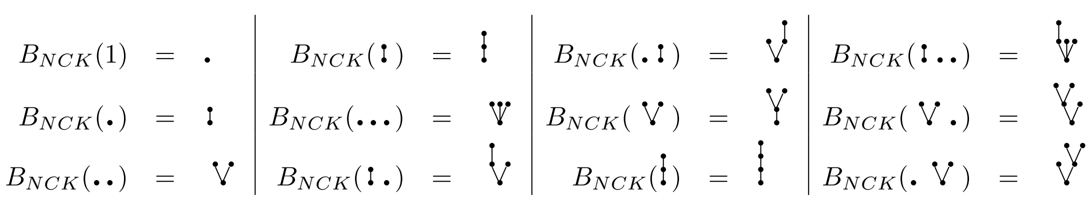
$$T_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2}, \qquad F_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1.1}$$

This gives :

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{NCK}}$	1	1	2	5	14	42	132	429	1430	4862	16796

As in the commutative case, we define the operator $B_{NCK} : \mathbf{H}_{NCK} \rightarrow \mathbf{H}_{NCK}$, which associates, to a planar forest $F \in \mathbf{H}_{NCK}$, the tree obtained by grafting the roots of the trees of F (from left to right) on a common root.

Examples.



We have a noncommutative version of theorem 1 (see [Moe01]) :

Theorem 2 *Let A be an algebra and let $L : A \rightarrow A$ be a linear map. Then there exists a unique morphism $\phi : \mathbf{H}_{NCK} \rightarrow A$, such that $\phi \circ B_{NCK} = L \circ \phi$.*

As in the commutative case, $\Delta_{\mathbf{H}_{NCK}} : \mathbf{H}_{NCK} \rightarrow \mathbf{H}_{NCK} \otimes \mathbf{H}_{NCK}$ is the unique algebra morphism such that

$$\Delta_{\mathbf{H}_{NCK}} \circ B_{NCK} = B_{NCK} \otimes 1 + (Id \otimes B_{NCK}) \circ \Delta_{\mathbf{H}_{NCK}}.$$

1.3 Ordered and heap-ordered forests

1.3.1 Construction

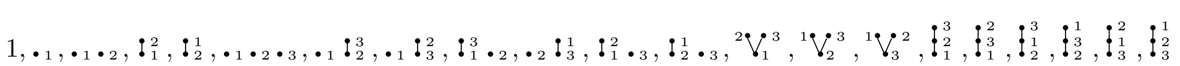
Definition 3 *An ordered forest F is a rooted forest F with a total order on $V(F)$. The set of ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_o}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_o}$ is denoted by $\mathbf{H}_o$.*

Remarks and notations. If F is an ordered forest, then there exits a unique increasing bijection $\sigma : V(F) \rightarrow \{1, \ldots, |F|_v\}$ for the total order on $V(F)$.

Reciprocally, if F is a rooted forest and if $\sigma : V(F) \rightarrow \{1, \ldots, |F|_v\}$ is a bijection, then σ defines a total order on $V(F)$ and F is an ordered forest.

Depending on the case, we shall denote an ordered forest by F or (F, σ).

Examples. Ordered forests of vertices degree ≤ 3 :



Let (F,σ^F) and (G,σ^G) be two ordered forests. Then the rooted forest FG is also an ordered forest (FG,σ^{FG}) where

$$\sigma^{FG}=\sigma^F\otimes\sigma^G:\left\{\begin{array}{rcl} V(F)\bigcup V(G) & \to & \{1,\ldots,|F|_v+|G|_v\}\\ a\in V(F) & \mapsto & \sigma^F(a)\\ a\in V(G) & \mapsto & \sigma^G(a)+|F|_v\,. \end{array}\right. \tag{1.2}$$

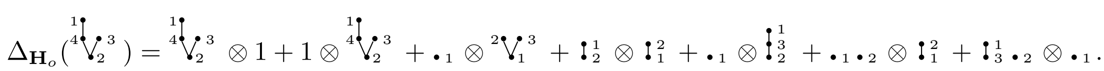
In other terms, we keep the initial order on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. This defines a noncommutative product on the set of ordered forests. For example, the product of $\bullet_1$ and $\mathbf{I}^2_1$ gives $\bullet_1\mathbf{I}^3_2=\mathbf{I}^3_2\bullet_1$, whereas the product of $\mathbf{I}^2_1$ and $\bullet_1$ gives $\mathbf{I}^2_1\bullet_3=\bullet_3\mathbf{I}^2_1$. This product is linearly extended to $\mathbf{H}_o$, which in this way becomes an algebra.

$\mathbf{H}_o$ is graded by the number of vertices and there is $(n+1)^{n-1}$ ordered forests in vertices degree n, see sequence A000272 of [Slo].

If F is an ordered forest, then any subforest G of F is also ordered : the total order on $V(G)$ is the restriction of the total order of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_o}:\mathbf{H}_o\to\mathbf{H}_o\otimes\mathbf{H}_o$ on $\mathbf{H}_o$ in the following way : for all $F\in\mathbb{F}_{\mathbf{H}_o}$,

$$\Delta_{\mathbf{H}_o}(F)=\sum_{\boldsymbol{v}\models V(F)} Lea_{\boldsymbol{v}}(F)\otimes Roo_{\boldsymbol{v}}(F).$$

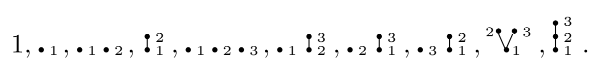
For example,



With this coproduct, $\mathbf{H}_o$ is a Hopf algebra.

Definition 4 *[GL90] A heap-ordered forest is an ordered forest F such that if $a,b\in V(F)$, $a\neq b$ and $a\twoheadrightarrow b$, then a is greater than b for the total order on $V(F)$. The set of heap-ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{ho}}$.*

Examples. Heap-ordered forests of vertices degree ≤ 3 :



Definition 5 *A linear order on a nonempty rooted forest F is a bijective map $\sigma:V(F)\to\{1,\ldots,|F|_v\}$ such that if $a,b\in V(F)$ and $a\twoheadrightarrow b$, then $\sigma(a)\geq\sigma(b)$. We denote by $\mathcal{O}(F)$ the set of linear orders on the nonempty rooted forest F.*

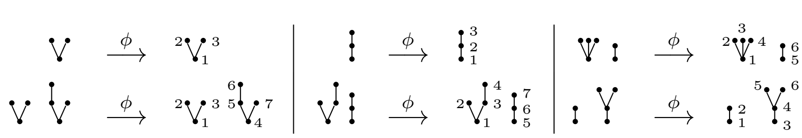
Remarks. If (F,σ) is a heap-ordered forest, then the increasing bijection $\sigma:V(F)\to\{1,\ldots,|F|_v\}$ is a linear order on F. Reciprocally, if F is a rooted forest and $\sigma\in\mathcal{O}(F)$, then σ defines a total order on $V(F)$ such that (F,σ) is a heap-ordered forest.

If F and G are two heap-ordered forests, then FG is an ordered forest with (1.2) and also a heap-ordered forest. Moreover, any subforest G of a heap-ordered forest F is also a heap-ordered forest by restriction on $V(G)$ of the total order of $V(F)$. So the subspace $\mathbf{H}_{ho}$ of $\mathbf{H}_o$ generated by the heap-ordered forests is a graded Hopf subalgebra of $\mathbf{H}_o$.

The number of heap-ordered forests of vertices degree n is $n!$, see sequence A000142 of [Slo].

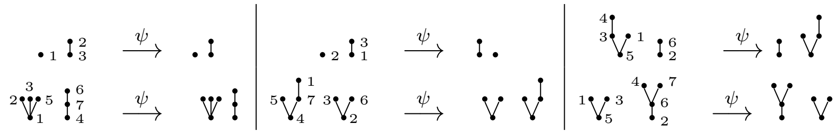
Remarks.

1. A planar forest can be considered as an ordered forest by ordering its vertices in the "north-west" direction, that is to say from bottom to top and from left to right (this is the order defined in [Foi02a] or given by the Depth First Search algorithm). This defines an algebra morphism $\phi:\mathbf{H}_{NCK}\to\mathbf{H}_o$. For example :



(1.3)

2. Reciprocally, an ordered forest is also planar, by restriction of the total order to the subsets of vertices formed by the roots or $\{w \in V(\mathbb{F}) \mid w \to v\}$. This defines an algebra morphism $\psi : \mathbf{H}_o \to \mathbf{H}_{NCK}$. For example :



(1.4)

Note that $\psi \circ \phi = Id_{\mathbf{H}_{NCK}}$ therefore ψ is surjective and ϕ is injective. ψ and ϕ are not bijective (by considering the dimensions).

Moreover ϕ is a Hopf algebra morphism and its image is included in $\mathbf{H}_{ho}$. ψ is not a Hopf algebra morphism : in the expression of $(\psi \otimes \psi) \circ \Delta_{\mathbf{H}_o}(\ )$ we have the tensor $\ \otimes\ $ and in the expression of $\Delta_{\mathbf{H}_{NCK}} \circ \psi(\ )$ we have the different tensor $\ \otimes\ $. By cons, the restriction of ψ to $\mathbf{H}_{ho}$ is a Hopf algebra morphism.

In the following, we consider $\mathbf{H}_{NCK}$ as a Hopf subalgebra of $\mathbf{H}_{ho}$ and $\mathbf{H}_o$.

1.3.2 Hopf algebra of permutations

Notations.

1. Let k, l be integers. A (k,l)-*shuffle* is a permutation ζ of $\{1, \ldots, k+l\}$, such that $\zeta(1) < \ldots < \zeta(k)$ and $\zeta(k+1) < \ldots < \zeta(k+l)$. The set of (k,l)-shuffles will be denoted by $Sh(k,l)$.
2. We represent a permutation $\sigma \in \Sigma_n$ by the word $(\sigma(1) \ldots \sigma(n))$.
   For example, $Sh(2,1) = \{(123),\ (132),\ (231)\}$.

Remark. For any integer k, l, any permutation $\sigma \in \Sigma_{k+l}$ can be uniquely written as $\epsilon \circ (\sigma_1 \otimes \sigma_2)$, where $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in Sh(k,l)$. Similarly, considering the inverses, any permutation $\tau \in \Sigma_{k+l}$ can be uniquely written as $(\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, where $\tau_1 \in \Sigma_k$, $\tau_2 \in \Sigma_l$, and $\zeta \in Sh(k,l)$. Note that, whereas ϵ renames the numbers of each lists $(\sigma(1), \ldots, \sigma(k)), (\sigma(k+1), \ldots, \sigma(k+l))$ without changing their orderings, ζ^{-1} shuffles the lists $(\tau(1), \ldots, \tau(k)), (\tau(k+1), \ldots, \tau(k+l))$. For instance, if $k = 4$, $l = 3$ and $\sigma = (5172436)$ then

- $\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)$, with $\epsilon = (1257346) \in Sh(4,3)$, $\sigma_1 = (3142) \in \Sigma_4$ and $\sigma_2 = (213) \in \Sigma_3$,
- $\sigma = (\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, with $\tau_1 = (1243) \in \Sigma_4$, $\tau_2 = (132) \in \Sigma_3$ and $\zeta = (2456137) \in Sh(4,3)$.

We here briefly recall the construction of the Hopf algebra **FQSym** of free quasi-symmetric functions, also called the Malvenuto-Reutenauer Hopf algebra [DHT02, MR95]. As a vector space, a basis of **FQSym** is given by the disjoint union of the symmetric groups Σ_n, for all $n \geq 0$. By convention, the unique element of Σ_0 is denoted by 1. The product of **FQSym** is given, for $\sigma \in \Sigma_k$, $\tau \in \Sigma_l$, by :

$$\sigma.\tau = \sum_{\zeta \in Sh(k,l)} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example :

$$\begin{aligned}(123)(21) &= (12354) + (12534) + (15234) + (51234) + (12543) \\ &\quad + (15243) + (51243) + (15423) + (51423) + (54123).\end{aligned}$$

Let $\sigma \in \Sigma_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in \Sigma_k \times \Sigma_{n-k} \times Sh(k, n-k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of **FQSym** is then defined by :

$$\Delta_{\mathbf{FQSym}}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2) \\ \epsilon \in Sh(k,n-k), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then *standardizing* the two obtained words by the following process. If v is a words of length n whose the letters are distinct integers, then the standardizing of v, denoted $Std(v)$, is the word obtained by applying to the letters of v the unique increasing bijection to $\{1,\ldots,n\}$. For example :

$$\begin{array}{rcl}\Delta_{\mathbf{FQSym}}((41325)) & = & 1\otimes(41325)+Std(4)\otimes Std(1325)+Std(41)\otimes Std(325)\\ & & +Std(413)\otimes Std(25)+Std(4132)\otimes Std(5)+(41325)\otimes 1\\ & = & 1\otimes(41325)+(1)\otimes(1324)+(21)\otimes(213)\\ & & +(312)\otimes(12)+(4132)\otimes(1)+(41325)\otimes 1.\end{array}$$

Then $\mathbf{FQSym}$ is a Hopf algebra. It is graded, with $\mathbf{FQSym}(n)=Vect(\Sigma_n)$ for all $n\geq 0$.

It is also possible to give a decorated version of $\mathbf{FQSym}$. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated permutation is a couple (σ,d), where $\sigma\in\Sigma_n$ and d is a map from $\{1,\ldots,n\}$ to $\mathcal{D}$. A $\mathcal{D}$-decorated permutation is represented by two superposed words $\binom{a_1\ldots a_n}{v_1\ldots v_n}$, where $(a_1\ldots a_n)$ is the word representing σ and for all i, $v_i=d(a_i)$. The vector space $\mathbf{FQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated permutations is a Hopf algebra. For example, if $x,y,z,t\in\mathcal{D}$:

$$\begin{array}{rcl}\binom{213}{yxz}\cdot\binom{1}{t} & = & \binom{2134}{yxzt}+\binom{2143}{yxtz}+\binom{2413}{ytxz}+\binom{4213}{tyxz},\\ \Delta_{\mathbf{FQSym}^{\mathcal{D}}}\left(\binom{4321}{tzyx}\right) & = & \binom{4321}{tzyx}\otimes 1+\binom{321}{tzy}\otimes\binom{1}{x}+\binom{21}{tz}\otimes\binom{21}{yx}+\binom{1}{t}\otimes\binom{321}{zyx}+1\otimes\binom{4321}{tzyx}.\end{array}$$

In other words, if (σ,d) and (τ,d') are decorated permutations of respective degrees k and l :

$$(\sigma,d).(\tau,d')=\sum_{\zeta\in Sh(k,l)}((\sigma\otimes\tau)\circ\zeta^{-1},d\otimes d'),$$

where $d\otimes d'$ is defined by $(d\otimes d')(i)=d(i)$ if $1\leq i\leq k$ and $(d\otimes d')(k+j)=d'(j)$ if $1\leq j\leq l$. If (σ,d) is a decorated permutation of degree n :

$$\Delta_{\mathbf{FQSym}^{\mathcal{D}}}((\sigma,d))=\sum_{k=0}^{n}\sum_{\substack{\sigma=\epsilon\circ(\sigma_1\otimes\sigma_2)\\ \epsilon\in Sh(k,l),\sigma_1\in\Sigma_k,\sigma_2\in\Sigma_l}}(\sigma_1,d')\otimes(\sigma_2,d''),$$

where $d=(d'\otimes d'')\circ\epsilon^{-1}$.

In some sense, a $\mathcal{D}$-decorated permutation can be seen as a word with a total order on the set of its letters.

We can now define the shuffle Hopf algebra $\mathbf{Sh}^{\mathcal{D}}$ (see [Hof00, Reu93]). A $\mathcal{D}$-word is a finite sequence of elements taken in $\mathcal{D}$. It is graded by the degree of words, that is to say the number of their letters. As a vector space, $\mathbf{Sh}^{\mathcal{D}}$ is generated by the set of $\mathcal{D}$-words.

The surjective linear map from $\mathbf{FQSym}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$, sending the decorated permutation $\binom{a_1\ldots a_n}{v_1\ldots v_n}$ to the $\mathcal{D}$-word $(v_1\ldots v_n)$, define a Hopf algebra structure on $\mathbf{Sh}^{\mathcal{D}}$:

- The product $⧢$ of $\mathbf{Sh}^{\mathcal{D}}$ is given in the following way : if $(v_1\ldots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1}\ldots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1\ldots v_k)⧢(v_{k+1}\ldots v_{k+l})=\sum_{\zeta\in Sh(k,l)}v_{\zeta^{-1}(1)}\ldots v_{\zeta^{-1}(k+l)}.$$

- The coproduct $\Delta_{\mathbf{Sh}^{\mathcal{D}}}$ of $\mathbf{Sh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $w=(v_1\ldots v_n)$ by

$$\Delta_{\mathbf{Sh}^{\mathcal{D}}}(w)=\sum_{i=0}^{n}(v_1\ldots v_i)\otimes(v_{i+1}\ldots v_n).$$

Examples.

1. If $(v_1v_2v_3)$ and (v_4v_5) are two $\mathcal{D}$-words,

$$\begin{array}{rcl}(v_1v_2v_3)⧢(v_4v_5) & = & (v_1v_2v_3v_4v_5)+(v_1v_2v_4v_3v_5)+(v_1v_2v_4v_5v_3)+(v_1v_4v_2v_3v_5)\\ & & +(v_1v_4v_2v_5v_3)+(v_1v_4v_5v_2v_3)+(v_4v_1v_2v_3v_5)+(v_4v_1v_2v_5v_3)\\ & & +(v_4v_1v_5v_2v_3)+(v_4v_5v_1v_2v_3).\end{array}$$

2. If $(v_1v_2v_3v_4)$ is a $\mathcal{D}$-word,

$$\begin{array}{rcl}\Delta_{\mathbf{Sh}^{\mathcal{D}}}((v_1v_2v_3v_4)) & = & (v_1v_2v_3v_4)\otimes 1+(v_1v_2v_3)\otimes(v_4)+(v_1v_2)\otimes(v_3v_4)\\ & & +(v_1)\otimes(v_2v_3v_4)+1\otimes(v_1v_2v_3v_4).\end{array}$$

There is a link between the algebras $\mathbf{H}_o$, $\mathbf{H}_{ho}$ and **FQSym** given by the following result (see [FU10]) :

Proposition 6 *1. Let $n \geq 0$. For all $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_o}$, let $\mathbb{S}_F$ be the set of permutations $\theta \in \Sigma_n$ such that for all $a,b \in V(F)$, $(a \twoheadrightarrow b) \Rightarrow (\theta^{-1}(\sigma(a)) \leq \theta^{-1}(\sigma(b)))$. Let us define :*

$$\Theta : \left\{ \begin{array}{rcl} \mathbf{H}_o & \rightarrow & \mathbf{FQSym} \\ F \in \mathbb{F}_{\mathbf{H}_o} & \mapsto & \displaystyle\sum_{\theta \in \mathbb{S}_F} \theta. \end{array} \right.$$

Then $\Theta : \mathbf{H}_o \rightarrow \mathbf{FQSym}$ is a Hopf algebra morphism, homogeneous of degree 0.

2. The restriction of Θ to $\mathbf{H}_{ho}$ is an isomorphism of graded Hopf algebras.

Chapitre 2

Preordered forests and Hopf algebras of contractions

Introduction

Nous introduisons dans ce chapitre la notion de forêts préordonnées. Un préordre est une relation binaire réflexive et transitive. Une forêt préordonnée est une forêt enracinée avec un préordre total sur ces sommets. Nous prouvons que l'algèbre des forêts préordonnées $\mathbf{H}_{po}$ est une algèbre de Hopf pour le coproduit de coupes. Avec une condition de croissance, on définit l'algèbre des forêts préordonnées en tas $\mathbf{H}_{hpo}$ et on prouve que $\mathbf{H}_{hpo}$ est une sous-algèbre de Hopf de $\mathbf{H}_{po}$.

Dans [NT06], J.-C. Novelli et J.-Y. Thibon construisent une généralisation de $\mathbf{FQSym}$: l'algèbre de Hopf $\mathbf{WQSym}^*$ des mots tassés. Nous prouvons un résultat similaire à celui de L. Foissy et J. Unterberger (voir [FU10]) en remplaçant les forêts ordonnées par les forêts préordonnées et les fonctions quasi-symétriques par les mots tassés. Plus précisément, on prouve qu'il existe un morphisme d'algèbres de Hopf de $\mathbf{H}_{po}$ dans $\mathbf{WQSym}^*$. De plus, nous montrons que sa restriction à $\mathbf{H}_{hpo}$ est une injection d'algèbres de Hopf.

Ensuite, nous étudions un nouveau coproduit, appelé ici coproduit de contraction. Dans [CEFM11], D. Calaque, K. Ebrahimi-Fard et D. Manchon définissent ce coproduit dans un cas commutatif, sur un quotient $\mathbf{C}_{CK}$ de $\mathbf{H}_{CK}$ (voir aussi [MS11]). Nous donnons une version décorée $\mathbf{C}_{CK}^{\mathcal{D}}$ de $\mathbf{C}_{CK}$. On définit deux opérations $\curlyvee$ et $\triangleright$ sur l'espace vectoriel $\mathbf{T}_{CK}^{\mathcal{D}}$ de base les arbres de $\mathbf{C}_{CK}^{\mathcal{D}}$. Nous introduisons la notion d'algèbre commutative pré-Lie : $(A, \curlyvee, \triangleright)$ est une algèbre commutative pré-Lie si $(A, \curlyvee)$ est une algèbre commutative, si $(A, \triangleright)$ est une algèbre pré-Lie et si, pour tout $x, y, z \in A$,

$$x \triangleright (y \curlyvee z) = (x \triangleright y) \curlyvee z + (x \triangleright z) \curlyvee y.$$

On prouve alors que $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ est une algèbre commutative pré-Lie engendrée par les arbres $\overset{\bullet}{\underset{\bullet}{|}}{}^{d}$, $d \in \mathcal{D}$.

Nous construisons une version non commutative de $\mathbf{C}_{CK}$. Pour cela, nous considérons des quotients de $\mathbf{H}_{NCK}$, $\mathbf{H}_{ho}$, $\mathbf{H}_{o}$, $\mathbf{H}_{hpo}$, $\mathbf{H}_{po}$, notés respectivement $\mathbf{C}_{NCK}$, $\mathbf{C}_{ho}$, $\mathbf{C}_{o}$, $\mathbf{C}_{hpo}$, $\mathbf{C}_{po}$, et nous définissons sur ces quotients un coproduit de contraction. Nous prouvons que $\mathbf{C}_{ho}$, $\mathbf{C}_{o}$, $\mathbf{C}_{hpo}$, $\mathbf{C}_{po}$ sont des algèbres de Hopf et que $\mathbf{C}_{NCK}$ est un comodule à gauche sur l'algèbre de Hopf $\mathbf{C}_{ho}$.

Enfin, nous étudions les morphismes d'algèbres de Hopf de $\mathbf{H}_{CK}^{\mathcal{D}}$ ou $\mathbf{C}_{CK}^{\mathcal{D}}$ dans l'algèbre de Hopf $\mathbf{Sh}^{\mathcal{D}}$ des battages ou dans l'algèbre de Hopf $\mathbf{Csh}^{\mathcal{D}}$ des battages contractants (voir [Hof00]). Nous donnons une description combinatoire de ces morphismes dans chacun des cas. Pour les morphismes de $\mathbf{H}_{CK}^{\mathcal{D}}$ dans $\mathbf{Sh}^{\mathcal{D}}$ ou $\mathbf{Csh}^{\mathcal{D}}$, on remarque en particulier que le coproduit de contraction et les forêts préordonnées apparaissent naturellement.

Le chapitre est organisé comme suit : la première partie est consacrée à la définition des algèbres $\mathbf{H}_{po}$ et $\mathbf{H}_{hpo}$ des forêts préordonnées et préordonnées en tas et nous prouvons que ce sont des algèbres de Hopf. Nous rappelons la définition de l'algèbre de Hopf $\mathbf{WQSym}^*$ des mots tassés et nous construisons un morphisme d'algèbres de Hopf de $\mathbf{H}_{po}$ dans $\mathbf{WQSym}^*$. Dans une seconde partie, nous introduisons un nouveau coproduit, le coproduit de contraction. Nous décrivons le cas commutatif (...et nous étudions une opération d'insertion...). Nous donnons une version non commutative utilisant les forêts ordonnées et préordonnées. Nous étudions dans la dernière partie les morphismes d'algèbres de Hopf de $\mathbf{H}_{CK}^{\mathcal{D}}$ ou $\mathbf{C}_{CK}^{\mathcal{D}}$ dans $\mathbf{Sh}^{\mathcal{D}}$ ou $\mathbf{Csh}^{\mathcal{D}}$ et nous donnons une description combinatoire de ces morphismes dans chacun des cas.

2.1 Preordered forests

2.1.1 Preordered and heap-preordered forests

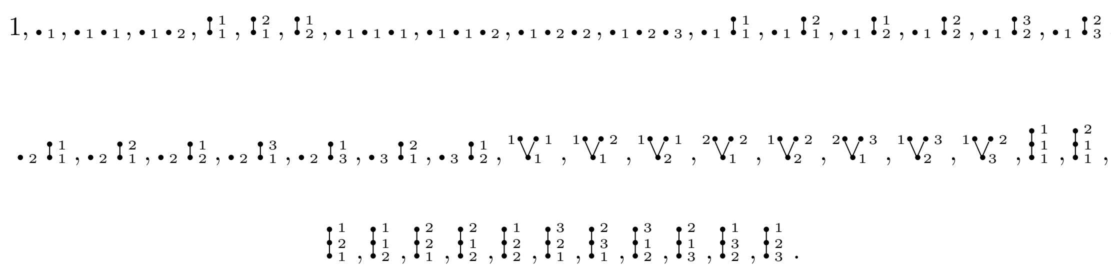
A preorder is a binary, reflexive and transitive relation. In particular, an antisymmetric preorder is an order. A preorder is total if two elements are always comparable. We introduce another version of ordered forests, the preordered forests.

Definition 7 *A preordered forest F is a rooted forest F with a total preorder on $V(F)$. The set of preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{po}}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{po}}$ is denoted by $\mathbf{H}_{po}$.*

Remarks and notations.

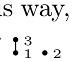
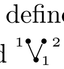
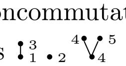
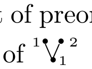
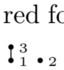
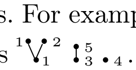
1. Let F be a preordered forest. We denote by $\leq$ the total preorder on $V(F)$. Remark that the antisymmetric relation "$x \leq y$ and $y \leq x$" is an equivalence relation denoted by $\mathcal{R}$ and the quotient set $V(F)/\mathcal{R}$ is totally ordered. We denote by q the cardinality of this quotient set. Let $\overline{\sigma}$ be the unique increasing map from $V(F)/\mathcal{R}$ to $\{1,\dots,q\}$. There exists a unique surjection $\sigma : V(F) \rightarrow \{1,\dots,q\}$, compatible with the equivalence $\mathcal{R}$, such that the induced map on $V(F)/\mathcal{R}$ is $\overline{\sigma}$. In the sequel, we shall note $q = \max(F)$ (and we have always $q \leq |F|_v$).
   Reciprocally, if F is a rooted forest and if $\sigma : V(F) \rightarrow \{1,\dots,q\}$ is a surjection, $q \leq |F|_v$, then σ define a total preorder on $V(F)$ and F is a preordered forest.
   As in the ordered case, we shall denote a preordered forest by F or (F,σ).
2. An ordered forest is also a preordered forest. Conversely, a preordered forest (F,σ) is an ordered forest if $|F|_v = \max(F)$.

Examples. Preordered forests of vertices degree ≤ 3 :



Let (F,σ^F) and (G,σ^G) be two preordered forests with $\sigma^F : V(F) \rightarrow \{1,\dots,q\}$, $\sigma^G : V(G) \rightarrow \{1,\dots,r\}$, $q = max(F)$ and $r = max(G)$. Then FG is also a preordered forest (FG,σ^{FG}) where

$$\sigma^{FG} = \sigma^F \otimes \sigma^G : \left\{ \begin{array}{rcl} V(F)\bigcup V(G) & \rightarrow & \{1,\dots,q+r\} \\ a \in V(F) & \mapsto & \sigma^F(a) \\ a \in V(G) & \mapsto & \sigma^G(a)+q. \end{array} \right. \tag{2.1}$$

In other terms, we keep the initial preorder on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. In this way, we define a noncommutative product on the set of preordered forests. For example, the product of  and  gives , whereas the product of  and  gives . Remark that, if F and G are two preordered forests, $\max(FG) = \max(F)+\max(G)$. This product is linearly extended to $\mathbf{H}_{po}$, which in this way becomes an algebra, gradued by the number of vertices.

Remark. The formula (2.1) on the preordered forests extends the formula (1.2) on the ordered forests.

We give some numerical values : if $f_n^{\mathbf{H}_{po}}$ is the number of preordered forests of vertices degree n,

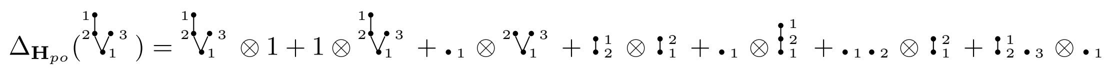
n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{po}}$	1	1	5	38	424	6284

These numerical values were calculated with the software Mupad.

If F is a preordered forest, then any subforest G of F is also preordered : the total preorder on $V(G)$ is the restriction of the total preorder of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_{po}} : \mathbf{H}_{po} \to \mathbf{H}_{po} \otimes \mathbf{H}_{po}$ on $\mathbf{H}_{po}$ in the following way : for all $F \in \mathbb{F}_{\mathbf{H}_{po}}$,

$$\Delta_{\mathbf{H}_{po}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

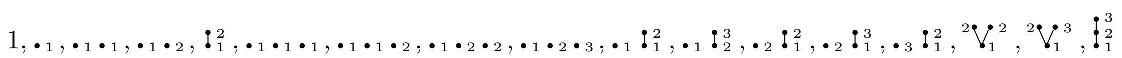
For example,



With this coproduct, $\mathbf{H}_{po}$ is a Hopf algebra. Remark that $\mathbf{H}_o$ is a Hopf subalgebra of $\mathbf{H}_{po}$.

Definition 8 *A heap-preordered forest is a preordered forest F such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$, then a is strictly greater than b for the total preorder on $V(F)$. The set of heap-preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{hpo}}$*

Examples. Heap-preordered forests of vertices degree ≤ 3 :



Definition 9 *Let F be a nonempty rooted forest and q an integer $\leq |F|_v$. A linear preorder is a surjection $\sigma : V(F) \to \{1, \dots, q\}$ such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$ then $\sigma(a) > \sigma(b)$. We denote by $\mathcal{O}_p(F)$ the set of linear preorders on the nonempty rooted forest F.*

Remarks. If (F, σ) is a heap-preordered forest, the surjection $\sigma : V(F) \to \{1, \dots, \max(F)\}$ is a linear preorder on F. Reciprocally, if F is a rooted forest and $\sigma \in \mathcal{O}_p(F)$, then σ define a total preorder on $V(F)$ such that (F, σ) is a heap-preordered forest.

If F and G are two heap-preordered forests, then FG is also heap-preordered. Moreover, any subforest G of a heap-preordered forest F is also a heap-preordered forest by restriction on $V(G)$ of the total preorder of $V(F)$. So the subspace $\mathbf{H}_{hpo}$ of $\mathbf{H}_{po}$ generated by the heap-preordered forests is a graded Hopf subalgebra of $\mathbf{H}_{po}$.

We give some numerical values : if $f_n^{\mathbf{H}_{hpo}}$ is the number of preordered forests of vertices degree n,

n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{hpo}}$	1	1	3	12	64	428

These numerical values were also calculated with the software Mupad.

We have the following diagram

$$\begin{tikzcd} \mathbf{H}_{NCK} \arrow[r, hook] & \mathbf{H}_{ho} \arrow[r, hook] \arrow[d, hook] & \mathbf{H}_{o} \arrow[d, hook] \\ & \mathbf{H}_{hpo} \arrow[r, hook] & \mathbf{H}_{po} \end{tikzcd} \tag{2.2}$$

where the arrows $\hookrightarrow$ are injective morphisms of Hopf algebras (for the cut coproduct).

2.1.2 Hopf algebra of packed words and quasi-shuffles

Recall the construction of the Hopf algebra $\mathbf{WQSym}^*$ of free packed words (see [NT06]).

Notations.

1. Let $n \geq 0$. We denote by $Surj_n$ the set of maps $\sigma : \{1, \dots, n\} \to \mathbb{N}^*$, such that $\sigma(\{1, \dots, n\}) = \{1, \dots, k\}$ for some $k \in \mathbb{N}$. In this case, k is the maximum of σ and is denoted by $\max(\sigma)$ and n is the length of σ. We represent the element σ of $Surj_n$ by the packed word $(\sigma(1) \dots \sigma(n))$.

2. Let k, l be two integers. A *(k,l)-surjective shuffle* is an element ϵ of $Surj_{k+l}$ such that $\epsilon(1) < \ldots < \epsilon(k)$ and $\epsilon(k+1) < \ldots < \epsilon(k+l)$. The set of (k,l)-surjective shuffles will be denoted by $SjSh(k,l)$. For example, $SjSh(2,1) = \{(121),(122),(123),(132),(231)\}$.

Let v be a word such that the letters occuring in v are integers $a_1 < a_2 < \ldots < a_k$. The *packing* of v, denoted by $pack(v)$, is the image of letters of v by the application $a_i \mapsto i$. For example, $pack((22539)) = (11324)$, $pack((831535)) = (421323)$.

Remark. Let k, l be two integers and $\sigma \in Surj_{k+l}$. We set $p_k = \max(pack((\sigma(1)\ldots\sigma(k))))$ and $q_k = \max(pack((\sigma(k+1)\ldots\sigma(k+l))))$. Then σ can be uniquely written as $\epsilon \circ (\sigma_1 \otimes \sigma_2)$, where $\sigma_1 \in Surj_k$, $\sigma_2 \in Surj_l$, and $\epsilon \in SjSh(p_k, q_k)$. For instance, if $k = 4$, $l = 3$ and $\sigma = (2311223)$ then $p_4 = 3$, $q_4 = 2$ and $\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)$ with $\epsilon = (12323) \in SjSh(3,2)$, $\sigma_1 = (2311) \in Surj_4$ and $\sigma_2 = (112) \in Surj_3$.

As a vector space, a basis of $\mathbf{WQSym}^*$ is given by the disjoint union of the sets $Surj_n$, for all $n \geq 0$. By convention, the unique element of $Surj_0$ is denoted by 1. The product of $\mathbf{WQSym}^*$ is given, for $\sigma \in Surj_k$ and $\tau \in Surj_l$, by :

$$\sigma.\tau = \sum_{\zeta \in Sh(k,l)} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other words, as in the **FQSym** case, the product of σ and τ is given by shifting the letters of the word representing τ by k, and summing all the possible shuffings of this word and of the word representing σ. For example :

$$\begin{array}{rcl}(112)(21) & = & (11243)+(11423)+(14123)+(41123)+(11432)+(14132)\\ & & +(41132)+(14312)+(41312)+(43112)\end{array}$$

Let $\sigma \in Surj_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in Surj_k \times Surj_{n-k} \times SjSh(p_k, q_k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of $\mathbf{WQSym}^*$ is then given by :

$$\Delta_{\mathbf{WQSym}^*}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)\\ \epsilon \in SjSh(p_k,q_k), \sigma_1 \in Surj_k, \sigma_2 \in Surj_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words. For example :

$$\begin{array}{rcl}\Delta_{\mathbf{WQSym}^*}((21132)) & = & 1 \otimes (21132) + pack((2)) \otimes pack((1132)) + pack((21)) \otimes pack((132))\\ & & +pack((211)) \otimes pack((32)) + pack((2113)) \otimes pack((2)) + (21132) \otimes 1\\ & = & 1 \otimes (21132) + (1) \otimes (1132) + (21) \otimes (132) + (211) \otimes (21)\\ & & +(2113) \otimes (1) + (21132) \otimes 1.\end{array}$$

Then $\mathbf{WQSym}^*$ is a graded Hopf algebra, with $\mathbf{WQSym}^*(n) = Surj_n$ for all $n \geq 0$. We give some numerical values : if $f_n^{\mathbf{WQSym}^*}$ is the number of packed words of length n, then

n	0	1	2	3	4	5	6	7
$f_n^{\mathbf{WQSym}^*}$	1	1	3	13	75	541	4683	47293

These is the sequence A000670 in [Slo].

$\mathbf{WQSym}^*$ is the gradued dual of $\mathbf{WQSym}$, described as follows. A basis of $\mathbf{WQSym}$ is given by the disjoint union of the sets $Surj_n$. The product of $\mathbf{WQSym}$ is given, for $\sigma \in Surj_k$, $\tau \in Surj_l$ by :

$$\sigma.\tau = \sum_{\substack{\gamma = \gamma_1 \cdots \gamma_{k+l}\\ pack(\gamma_1 \cdots \gamma_k) = \sigma,\ pack(\gamma_{k+1} \cdots \gamma_{k+l}) = \tau}} \gamma$$

In other terms, the product of σ and τ is given by shifting certain letters of the words representing σ and τ and then summing all concatenations of obtained words. For example :

$$\begin{array}{rcl}(112)(21) & = & (11221)+(11321)+(22321)+(33421)+(11231)+(22331)+(22431)\\ & & +(11232)+(11332)+(11432)+(22341)+(11342)+(11243)\end{array}$$

If $\sigma \in Surj_n$, then the coproduct of **WQSym** is given by :

$$\Delta_{\mathbf{WQSym}}(\sigma) = \sum_{0 \leq k \leq max(\sigma)} \sigma_{|[1,k]} \otimes pack(\sigma_{|[k+1,max(\sigma)]}),$$

where $\sigma_{|\mathcal{A}}$ is the subword obtained by tacking in σ the letters from the subset $\mathcal{A}$ of $[1, max(\sigma)]$. For example :

$$\begin{array}{rcl} \Delta_{\mathbf{WQSym}}((21312245)) & = & 1 \otimes (21312245) + (11) \otimes pack((232245)) + (21122) \otimes pack((345)) \\ & & +(213122) \otimes pack((45)) + (2131224) \otimes pack((5)) + (21312245) \otimes 1 \\ & = & 1 \otimes (21312245) + (11) \otimes (121134) + (21122) \otimes (123) \\ & & +(213122) \otimes (12) + (2131224) \otimes (1) + (21312245) \otimes 1. \end{array}$$

Then **WQSym** is a Hopf algebra.

We give a decorated version of **WQSym**. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated surjection is a couple (σ, d), where $\sigma \in Surj_n$ and d is a map from $\{1, \ldots, n\}$ to $\mathcal{D}$. As in the $\mathbf{FQSym}^{\mathcal{D}}$ case, we represent a $\mathcal{D}$-decorated surjection by two superposed words $\binom{a_1 \ldots a_n}{v_1 \ldots v_n}$, where $(a_1 \ldots a_n)$ is the packed word representing σ and for all i, $v_i = d(i)$. The vector space $\mathbf{WQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated surjections is a Hopf algebra. For example, if $x, y, z, t \in \mathcal{D}$:

$$\begin{array}{rcl} \binom{211}{yxz} \cdot \binom{1}{t} & = & \binom{2111}{yxzt} + \binom{2112}{yxzt} + \binom{2113}{yxzt} + \binom{3221}{yxzt} + \binom{3112}{yxzt}. \\ \Delta_{\mathbf{WQSym}^{\mathcal{D}}}\left(\binom{2113}{yxzt}\right) & = & \binom{2113}{yxzt} \otimes 1 + \binom{11}{xz} \otimes \binom{12}{yt} + \binom{211}{yxz} \otimes \binom{1}{t} + 1 \otimes \binom{2113}{yxzt}. \end{array}$$

In other words, if (σ, d) and (τ, d') are decorated surjections of respective degrees k and l :

$$(\sigma, d).(\tau, d') = \sum_{\substack{\gamma = \gamma_1 \ldots \gamma_{k+l} \\ pack(\gamma_1 \ldots \gamma_k) = \sigma,\ pack(\gamma_{k+1} \ldots \gamma_{k+l}) = \tau}} (\gamma, d \otimes d'),$$

where $d \otimes d'$ is defined by $(d \otimes d')(i) = d(i)$ if $1 \leq i \leq k$ and $(d \otimes d')(k+j) = d'(j)$ if $1 \leq j \leq l$. If (σ, d) is a decorated surjection of degree n :

$$\Delta_{\mathbf{WQSym}^{\mathcal{D}}}((\sigma, d)) = \sum_{0 \leq k \leq \max(\sigma)} (\sigma_{|[1,k]}, d') \otimes (\sigma_{|[k+1, \max(\sigma)]}, d'').$$

where d' and d take the same values on $\sigma^{-1}(\{1, \ldots, k\})$ and d'' and d take the same values on $\sigma^{-1}(\{k+1, \ldots, \max(\sigma)\})$.

In some sense, a $\mathcal{D}$-decorated surjection can be seen as a packed word with a preorder on the set of its letters.

Suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot, \cdot] : (a, b) \in \mathcal{D}^2 \to [ab] \in \mathcal{D}$. We define by induction $[\cdot, \cdot]^{(k)}$:

$$[\cdot, \cdot]^{(0)} = Id,\ [\cdot, \cdot]^{(1)} = [\cdot, \cdot] \text{ and } [\cdot, \cdot]^{(k)} = \left[\cdot, [\cdot, \cdot]^{(k-1)}\right].$$

We can now define the quasi-shuffle Hopf algebra $\mathbf{Csh}^{\mathcal{D}}$ (see [Hof00]). $\mathbf{Csh}^{\mathcal{D}}$ is, as a vector space, generated by the set of $\mathcal{D}$-words.

Let φ be the surjective linear map from $\mathbf{WQSym}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$ defined, for (σ, d) a decorated surjection of maximum k, by $\varphi((\sigma, d)) = (w_1 \ldots w_k)$ where $w_j = [d(i_1) \ldots d(i_p)]^{(p)}$ with $\sigma^{-1}(j) = \{i_1, \ldots, i_p\}$. For instance,

$$\varphi\left(\binom{2114324}{yxztvwu}\right) = ([xz]\ [yw]\ v\ [tu])$$

Notations. Let k, l be integers. A (k, l)-quasi-shuffle of type r is a surjective map $\zeta : \{1, \ldots, k+l\} \twoheadrightarrow \{1, \ldots, k+l-r\}$ such that

$$\begin{cases} \zeta(1) < \ldots < \zeta(k), \\ \zeta(k+1) < \ldots < \zeta(k+l). \end{cases}$$

Remark that $\zeta^{-1}(j)$ contains 1 or 2 elements for all $1 \leq j \leq k+l-r$. The set of (k, l)-quasi-shuffles of type r is denoted by $Csh(p, q, r)$. Remark that $Csh(k, l, 0) = Sh(k, l)$.

φ define a Hopf algebra structure on $\mathbf{Csh}^{\mathcal{D}}$:

- The product $\uplus$ of $\mathbf{Csh}^{\mathcal{D}}$ is given in the following way : if $(v_1 \dots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1} \dots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1 \dots v_k) \uplus (v_{k+1} \dots v_{k+l}) = \sum_{r \geq 0} \sum_{\zeta \in Csh(k,l,r)} (w_1 \dots w_{k+l-r}),$$

  where $w_j = v_i$ if $\zeta^{-1}(j) = \{i\}$ and $w_j = [v_{i_1} v_{i_2}]$ if $\zeta^{-1}(j) = \{i_1, i_2\}$.
- The coproduct $\Delta_{\mathbf{Csh}^{\mathcal{D}}}$ of $\mathbf{Csh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $v = (v_1 \dots v_n)$ by

$$\Delta_{\mathbf{Csh}^{\mathcal{D}}}(v) = \sum_{i=0}^{n} (v_1 \dots v_i) \otimes (v_{i+1} \dots v_n).$$

  This is the same coproduct as for $\mathbf{Sh}^{\mathcal{D}}$.

Example. If $(v_1 v_2)$ and $(v_3 v_4)$ are two $\mathcal{D}$-words,

$$\begin{aligned}(v_1 v_2) \uplus (v_3 v_4) &= (v_1 v_2 v_3 v_4) + (v_1 v_3 v_2 v_4) + (v_3 v_1 v_2 v_4) + (v_1 v_3 v_4 v_2) \\ &\quad + (v_3 v_1 v_4 v_2) + (v_3 v_4 v_1 v_2) + (v_1 [v_2 v_3] v_4) + ([v_1 v_3] v_2 v_4) \\ &\quad + (v_1 v_3 [v_2 v_4]) + (v_3 [v_1 v_4] v_2) + ([v_1 v_3] [v_2 v_4])\end{aligned}$$

2.1.3 A morphism from $\mathbf{H}_{po}$ to $\mathbf{WQSym}^*$

In this section, we give a similar result of proposition 6 in the preordered case.

Definition 10 *Let (F, σ) be a nonempty preordered forest of vertices degree n and $\tau \in Surj_n$. Then we denote by $\mathbb{S}^{\tau}_{(F,\sigma)}$ the set of bijective maps $\varphi : V(F) \to \{1, \dots, n\}$ such that :*

1. *if $v \in V(F)$, then $\sigma(v) = \tau(\varphi(v))$,*
2. *if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.*

Remark.

1. If $\max(F) \neq \max(\tau)$, then $\mathbb{S}^{\tau}_{(F,\sigma)} = \emptyset$.
2. Let $F, G \in \mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v = k$, $|G|_v = l$. If $\varphi_1 : V(F) \to \{1, \dots, k\}$ and $\varphi_2 : V(G) \to \{1, \dots, l\}$ are two bijective maps and $\zeta \in Sh(k, l)$, then $\zeta \circ (\varphi_1 \otimes \varphi_2) : V(FG) \to \{1, \dots, k + l\}$, where $\varphi_1 \otimes \varphi_2$ is defined in formula (1.2), is also a bijective map. Similary, considering a bijective map $\varphi : V(FG) \to \{1, \dots, k + l\}$ and $\zeta \in Sh(k, l)$. Then φ can be uniquely written as $\zeta \circ (\varphi_1 \otimes \varphi_2)$, where $\varphi_1 : V(F) \to \{1, \dots, k\}$ and $\varphi_2 : V(G) \to \{1, \dots, l\}$ are two bijective maps.

Theorem 11 *Let us define :*

$$\Phi : \left\{ \begin{array}{rcl} \mathbf{H}_{po} & \to & \mathbf{WQSym}^* \\ (F, \sigma) \in \mathbb{F}_{\mathbf{H}_{po}} & \mapsto & \displaystyle\sum_{\tau \in Surj_{|F|_v}} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau. \end{array} \right. \tag{2.3}$$

Then $\Phi : \mathbf{H}_{po} \to \mathbf{WQSym}^*$ is a Hopf algebra morphism, homogeneous of degree 0.

Examples.
- In vertices degree 1 : $\Phi(\bullet_1) = (1)$.
- In vertices degree 2 :

$$\Phi(\bullet_1 \bullet_1) = 2(11), \qquad \Phi(\bullet_1 \bullet_2) = (12) + (21), \qquad \Phi\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\,{}^{a}_{b}\right) = (ab).$$

- In vertices degree 3 :

$$\begin{array}{rclrcl} \Phi(\bullet_1 \bullet_1 \bullet_1) &=& 6(111) & \Phi\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\,{}^{2}_{1}\ \bullet_2\right) &=& (212) + 2(221) \\ \Phi\left({}^{2}\mathsf{V}^{1}_{2}\right) &=& (122) + (212) & \Phi\left(\bullet_2\ \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\,{}^{1}_{3}\right) &=& (213) + (123) + (132) \\ \Phi\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\\ |\\ \bullet\end{smallmatrix}\,\begin{smallmatrix}2\\ 3\\ 1\end{smallmatrix}\right) &=& (231) & \Phi\left(\bullet_1\ \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\,{}^{2}_{3}\right) &=& (123) + (213) + (231) \\ \Phi\left({}^{2}\mathsf{V}^{2}_{1}\right) &=& 2(221) & \Phi(\bullet_1 \bullet_1 \bullet_2) &=& 2\,[(112) + (121) + (211)] \end{array}$$

Proof. Obviously, Φ is homogeneous of degree 0. Let $(F,\sigma^F),(G,\sigma^G)\in\mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v=k$, $|G|_v=l$ and $\tau\in Surj_{k+l}$. τ can be uniquely written as $\tau=(\tau_1\otimes\tau_2)\circ\zeta^{-1}$ with $\tau_1\in Surj_k$, $\tau_2\in Surj_l$ and $\zeta\in Sh(k,l)$.

Let $\varphi\in\mathbb{S}^{\tau}_{(FG,\sigma^{FG})}$. Then φ can be uniquely written as $\zeta\circ(\varphi_1\otimes\varphi_2)$ with $\varphi_1:V(F)\to\{1,\ldots,k\}$ and $\varphi_2:V(G)\to\{1,\ldots,l\}$ two bijective maps.

1. (a) If $v\in V(F)$, then
$$\sigma^F(v)=\sigma^{FG}(v)=\tau(\varphi(v))=(\tau_1\otimes\tau_2)\circ\zeta^{-1}\circ\zeta\circ(\varphi_1\otimes\varphi_2)(v)=\tau_1(\varphi_1(v)).$$
Note that with this equality, we also have that $\max(F)=\max(\tau_1)$.

   (b) If $v\in V(G)$, then
$$\begin{array}{rcl}\sigma^G(v)+\max(F)&=&\sigma^{FG}(v)=\tau(\varphi(v))=(\tau_1\otimes\tau_2)\circ\zeta^{-1}\circ\zeta\circ(\varphi_1\otimes\varphi_2)(v)\\&=&\tau_2(\varphi_2(v))+\max(\tau_1).\end{array}$$
As $\max(F)=\max(\tau_1)$, $\sigma^G(v)=\tau_2(\varphi_2(v))$.

2. (a) If $v'\twoheadrightarrow v$ in F, then $v'\twoheadrightarrow v$ in FG, so :
$$\begin{array}{rcl}\varphi(v)&\geq&\varphi(v')\\ \zeta\circ(\varphi_1\otimes\varphi_2)(v)&\geq&\zeta\circ(\varphi_1\otimes\varphi_2)(v')\\ \zeta\,(\varphi_1(v))&\geq&\zeta\,(\varphi_1(v'))\\ \varphi_1(v)&\geq&\varphi_1(v'),\end{array}$$
as ζ is increasing on $\{1,\ldots,k\}$.

   (b) If $v'\twoheadrightarrow v$ in G, then $v'\twoheadrightarrow v$ in FG, so :
$$\begin{array}{rcl}\varphi(v)&\geq&\varphi(v')\\ \zeta\circ(\varphi_1\otimes\varphi_2)(v)&\geq&\zeta\circ(\varphi_1\otimes\varphi_2)(v')\\ \zeta\,(k+\varphi_2(v))&\geq&\zeta\,(k+\varphi_2(v'))\\ \varphi_2(v)&\geq&\varphi_2(v'),\end{array}$$
as ζ is increasing on $\{k+1,\ldots,k+l\}$.

So $\varphi_1\in\mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2\in\mathbb{S}^{\tau_2}_{(G,\sigma^G)}$.

Conversely, if $\varphi=\zeta\circ(\varphi_1\otimes\varphi_2)$, with $\varphi_1\in\mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2\in\mathbb{S}^{\tau_2}_{(G,\sigma^G)}$, the same computations shows that $\varphi\in\mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})}$.

So
$$\mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})})=\mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)})\times\mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})$$
and
$$\begin{array}{rcl}\Phi((FG,\sigma^{FG}))&=&\displaystyle\sum_{\tau\in Surj_{k+l}}\mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})})\;\tau\\&=&\displaystyle\sum_{\zeta\in Sh(k,l)}\sum_{\tau_1\in Surj_k}\sum_{\tau_2\in Surj_l}\mathrm{card}(\mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})})\;(\tau_1\otimes\tau_2)\circ\zeta^{-1}\\&=&\displaystyle\sum_{\zeta\in Sh(k,l)}\sum_{\tau_1\in Surj_k}\sum_{\tau_2\in Surj_l}\mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)})\times\mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\;(\tau_1\otimes\tau_2)\circ\zeta^{-1}\\&=&\displaystyle\left(\sum_{\tau_1\in Surj_k}\mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)})\;\tau_1\right)\left(\sum_{\tau_2\in Surj_l}\mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\;\tau_2\right)\\&=&\Phi((F,\sigma^F))\Phi((G,\sigma^G)).\end{array}$$
So Φ is an algebra morphism.

Let $(F,\sigma)\in\mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest such that $|F|_v=n$ and let $\boldsymbol{v}$ be an admissible cut of F. We obtain two preordered forests $(Lea_{\boldsymbol{v}}(F),\sigma_1)$ and $(Roo_{\boldsymbol{v}}(F),\sigma_2)$. We set $k=|Lea_{\boldsymbol{v}}(F)|_v$ and $l=|Roo_{\boldsymbol{v}}(F)|_v$.

Let $\tau_1\in Surj_k$, $\tau_2\in Surj_l$ and $\varphi_1\in\mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_2\in\mathbb{S}^{\tau_2}_{(Roo_{\boldsymbol{v}}(F),\sigma_2)}$. We set $\varphi=\varphi_1\otimes\varphi_2$ and we define τ by $\tau=\sigma\circ\varphi^{-1}$. $\tau\in Surj_n$ and $\max(\tau)=\max(F)$. Let us show that $\varphi\in\mathbb{S}^{\tau}_{(F,\sigma)}$.

1. By definition, $\tau = \sigma \circ \varphi^{-1}$. So $\sigma(v) = \tau(\varphi(v))$ for all $v \in V(F)$.
2. If $v' \twoheadrightarrow v$ in F, then three cases are possible :
   (a) v and v' belong to $V(Lea_{\boldsymbol{v}}(F))$. As $\varphi_1 \in \mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_1(v) \geq \varphi_1(v')$. Then $\varphi(v) = (\varphi_1 \otimes \varphi_2)(v) = \varphi_1(v) \geq \varphi_1(v') = (\varphi_1 \otimes \varphi_2)(v') = \varphi(v')$.
   (b) v and v' belong to $V(Roo_{\boldsymbol{v}}(F))$. As $\varphi_2 \in \mathbb{S}^{\tau_2}_{(Lea_{\boldsymbol{v}}(F),\sigma_2)}$, $\varphi_2(v) \geq \varphi_2(v')$. Then $\varphi(v) = (\varphi_1 \otimes \varphi_2)(v) = \varphi_2(v) + k \geq \varphi_2(v') + k = (\varphi_1 \otimes \varphi_2)(v') = \varphi(v')$.
   (c) v' belong to $V(Lea_{\boldsymbol{v}}(F))$ and v belong to $V(Roo_{\boldsymbol{v}}(F))$. Then $\varphi(v') = (\varphi_1 \otimes \varphi_2)(v') = \varphi_1(v') \in \{1, \ldots, k\}$ and $\varphi(v) = (\varphi_1 \otimes \varphi_2)(v) = \varphi_2(v) + k \in \{k+1, \ldots, k+l\}$. So $\varphi(v) > \varphi(v')$.

   In any case, $\varphi(v) \geq \varphi(v')$.

Conversely, let $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest of vertices degree n, $\tau \in Surj_n$ and $\varphi \in \mathbb{S}^{\tau}_{(F,\sigma)}$. Let $k \in \{0, \ldots, n\}$ be an integer.

We set $\tau_1^{(k)}$ and $\tau_2^{(k)}$ the words obtained by cutting the word representing τ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words.

Moreover, we define $\boldsymbol{v}$ a subset of $\varphi^{-1}(\{1, \ldots, k\})$ such that $v \not\twoheadrightarrow w$ for any couple (v,w) of two different elements of $\boldsymbol{v}$. Then $\boldsymbol{v} \models V(F)$ and we considere the two preordered forests $(Lea_{\boldsymbol{v}}(F), \sigma_1^{(k)})$ and $(Roo_{\boldsymbol{v}}(F), \sigma_2^{(k)})$. Remark that, with the second point of definition 10, $V(Lea_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{1, \ldots, k\})$ and $V(Roo_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{k+1, \ldots, n\})$.

We set $\varphi_1^{(k)} : v \in V(Lea_{\boldsymbol{v}}(F)) \to \varphi(v) \in \{1, \ldots, k\}$ and $\varphi_2^{(k)} : v \in V(Roo_{\boldsymbol{v}}(F)) \to \varphi(v) - k \in \{1, \ldots, n-k\}$. Thus $\varphi = \varphi_1^{(k)} \otimes \varphi_2^{(k)}$.

Let us prove that $\varphi_1^{(k)} \in \mathbb{S}^{\tau_1^{(k)}}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})}$ and $\varphi_2^{(k)} \in \mathbb{S}^{\tau_2^{(k)}}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})}$.

1. (a) If $v \in V(Lea_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_1^{(k)}(v) \in \{1, \ldots, k\}$ and then
$$\sigma_1^{(k)}(v) = pack \circ \sigma(v) = pack \circ \tau \circ \varphi(v) = \tau_1^{(k)} \circ \varphi_1^{(k)}(v).$$
   (b) If $v \in V(Roo_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_2^{(k)}(v) + k \in \{k+1, \ldots, n\}$ and then
$$\sigma_2^{(k)}(v) = pack \circ \sigma(v) = pack \circ \tau \circ \varphi(v) = \tau_2^{(k)} \circ \varphi_2^{(k)}(v).$$
2. (a) If $v' \twoheadrightarrow v$ in $Lea_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_1^{(k)}(v) = \varphi(v) \geq \varphi(v') = \varphi_2^{(k)}(v')$.
   (b) If $v' \twoheadrightarrow v$ in $Roo_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_2^{(k)}(v) = \varphi(v) - k \geq \varphi(v') - k = \varphi_2^{(k)}(v')$.

Hence, there is a bijection :
$$\left\{ \begin{array}{rcl} \mathbb{S}^{\tau}_{(F,\sigma)} \times \{0, \ldots, |F|_v\} & \to & \displaystyle\bigsqcup_{\boldsymbol{v} \models V(F)} \mathbb{S}^{\tau_1^{(k)}}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})} \times \mathbb{S}^{\tau_2^{(k)}}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})} \\ (\varphi, k) & \mapsto & \left(\varphi_1^{(k)}, \varphi_2^{(k)}\right). \end{array} \right.$$

Finally,
$$\begin{aligned} & \Delta_{\mathbf{WQSym}^*} \circ \Phi((F,\sigma)) \\ = & \sum_{\tau \in Surj_{|F|_v}} \sum_{0 \leq k \leq n} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau_1^{(k)} \otimes \tau_2^{(k)} \\ = & \sum_{\boldsymbol{v} \models V(F)} \sum_{\tau_1 \in Surj_{|Lea_{\boldsymbol{v}}(F)|_v}} \sum_{\tau_2 \in Surj_{|Roo_{\boldsymbol{v}}(F)|_v}} \mathrm{card}(\mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})})\ \tau_1 \otimes \mathrm{card}(\mathbb{S}^{\tau_2}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})})\ \tau_2 \\ = & (\Phi \otimes \Phi) \circ \Delta_{\mathbf{H}_{po}}. \end{aligned}$$

So Φ is a coalgebra morphism. □

Theorem 12 *The restriction of Φ defined in formula (2.3) to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras.*

Remark. The restriction of Φ to $\mathbf{H}_{hpo}$ is not bijective (by comparing the dimensions of $\mathbf{H}_{hpo}$ and $\mathbf{WQSym}^*$ in small degrees).

Proof. We introduce a lexicographic order on the words with letters $\in \mathbb{N}^*$. Let $u = (u_1 \ldots u_k)$ and $v = (v_1 \ldots v_l)$ be two words. Then

– if $u_k = v_k, u_{k-1} = v_{k-1}, \ldots, u_{i+1} = v_{i+1}$ and $u_i > v_i$ (resp. $u_i < v_i$) with $i \in \{1, \ldots, \min(k,l)\}$, then $u > v$ (resp. $u < v$),
– if $u_i = v_i$ for all $i \in \{1, \ldots, \min(k,l)\}$ and if $k > l$ (resp. $k < l$) then $u > v$ (resp. $u < v$).

For example,

$$(541) < (22), \quad (433) < (533), \quad (5362) < (72), \quad (8225) < (1327), \quad (215) < (1215).$$

If u and v are two words, we denote by uv the concatenation of u and v.

In this proof, if (F, σ) is a preordered forest, we consider F as a decorated forest where the vertices are decorated by integers. Consider

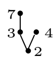
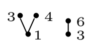
$$\mathbb{F} = \{(F,d) \ | \ F \in \mathbb{F}_{\mathbf{H}_{CK}}, d : V(F) \to \mathbb{N}^* \text{ such that if } v \to w \text{ then } d(v) > d(w)\}$$

the set of forests with their vertices decorated by nonzero integers and with an increasing condition.

Let $(F,d) \in \mathbb{F}$ be a forest of vertices degree n and if $u = (u_1 \ldots u_n)$ is a word of length n with $u_i \in \mathbb{N}^*$. In the same way that definition 10, we define $\mathbb{S}^u_{(F,d)}$ as the set of bijective maps $\varphi : V(F) \to \{1, \ldots, n\}$ such that :

1. if $v \in V(F)$, then $d(v) = u_{\varphi(v)}$,
2. if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.

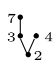
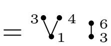
For example,

– if $(F,d) =$  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are $(7342), (7432), (4732)$.
– if $(F,d) =$  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are

$$\begin{array}{l}(43163), (43613), (46313), (64313), (43631), (46331), (64331), (63431),\\ (34163), (34613), (36413), (63413), (34631), (36431), (36341), (63341).\end{array}$$

Let (F,d) be a forest of $\mathbb{F}$. Then we set

$$m((F,d)) = \max\left(\{u \ | \ \mathbb{S}^u_{(F,d)} \neq \emptyset\}\right).$$

For example, for $(F,d) =$  $\in \mathbb{F}$, $m((F,d)) = (7342)$ and for $(F,d) =$  $\in \mathbb{F}$, $m((F,d)) = (34163)$.

If $(F,d) \in \mathbb{F}$ is the empty tree, $m((F,\sigma)) = 1$. Let $(F,d) \in \mathbb{F}$ be a nonempty tree of vertices degree n. We denote by (G,d') the forest of $\mathbb{F}$ obtained by deleting the root of F. Then, if $m((F,d)) = (u_1 \ldots u_n)$, we have $m((G,d')) = (u_1 \ldots u_{n-1})$ and $u_n = d(R_F)$ the decoration of the root of F. Let (F,d) be a forest of vertices degree n, (F,d) is the disjoint union of trees with their vertices decorated by nonzero integers $(F_1,d_1), \ldots, (F_k,d_k)$ ordered such that $m((F_1,d_1)) \leq \ldots \leq m((F_k,d_k))$. Then $m((F,d)) = m((F_1,d_1)) \ldots m((F_k,d_k))$:

– By definition, $\mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)} \neq \emptyset$ and if $\varphi_i \in \mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)}$ then $\varphi : V(F) \to \{1, \ldots, n\}$, defined for all $1 \leq i \leq k$ and $v \in V(F_i)$ by $\varphi(v) = \varphi_i(v)$, is an element of $\mathbb{S}^{m((F_1,d_1))\ldots m((F_k,d_k))}_{(F,d)}$ and $\mathbb{S}^{m((F_1,d_1))\ldots m((F_k,d_k))}_{(F,d)} \neq \emptyset$. So $m((F,d)) \geq m((F_1,d_1)) \ldots m((F_k,d_k))$.
– If $\mathbb{S}^u_{(F,d)} \neq \emptyset$, u is the shuffle of $u_1, \ldots, u_k$ such that $\mathbb{S}^{u_i}_{(F_i,d_i)} \neq \emptyset$ (see the proof of theorem 11). In particular, $u_i \leq m((F_i,d_i))$, so $u \leq m((F_1,d_1)) \ldots m((F_k,d_k))$. Thus we have $m((F,d)) \leq m((F_1,d_1)) \ldots m((F_k,d_k))$.

Let $(F,d) \in \mathbb{F}$ be a forest of vertices degree n and $m((F,d)) = (u_1 \ldots u_n)$. Let i_1 be the smallest index such that $u_1, \ldots, u_{i_1 - 1} > u_{i_1}$ and, for all $j > i_1$, $u_{i_1} \leq u_j$. By construction, there exists a connected component (F_1,d_1) of (F,d) such that $m((F_1,d_1)) = (u_1 \ldots u_{i_1})$. Consider the word $(u_{i_1+1} \ldots u_n)$. Let $i_2 > i_1$ be the smallest index such that $u_{i_1+1}, \ldots, u_{i_2-1} > u_{i_2}$ and, for all $j > i_2$, $u_{i_2} \leq u_j$. Then there exists a connected component (F_2,d_2) (different from (F_1,d_1)) such that $m((F_2,d_2)) = (u_{i_1+1} \ldots u_{i_2})$. In the same way, we construct $i_3, \ldots, i_k$ and $(F_3,d_3), \ldots, (F_k,d_k)$. Then $m((F,d)) = m((F_1,d_1)) \ldots m((F_k,d_k))$

Let us prove that m is injective on $\mathbb{F}$ by induction on the vertices degree. If (F,d) is the empty tree, it is obvious. Let (F,d) be a nonempty forest of $\mathbb{F}$ of vertices degree n.

- If (F,d) is a tree, $m((F,d)) = (u_1 \dots u_{n-1}u_n)$ with $u_n = d(R_F)$ the decoration of the root of F. Let (G,d') be the forest of $\mathbb{F}$ obtained by deleting the root of F. Then $m((G,d')) = (u_1 \dots u_{n-1})$. By induction hypothesis, (G,d') is the unique forest of $\mathbb{F}$ such that $m((G,d')) = (u_1 \dots u_{n-1})$. So (F,d) is also the unique forest of $\mathbb{F}$ such that $m((F,\sigma)) = (u_1 \dots u_{n-1}d(R_F))$.
- If (F,d) is not a tree, then (F,d) is the product of trees $(F_1,d_1),\dots,(F_k,d_k)$ of $\mathbb{F}$ ordered such that $m((F_1,d_1)) \leq \dots \leq m((F_k,d_k))$. So $m((F,d)) = m((F_1,d_1))\dots m((F_k,d_k))$. By induction hypothesis, for all $1 \leq i \leq k$, (F_i,d_i) is the unique tree of $\mathbb{F}$ such that its image by m is $m((F_i,d_i))$. So the product (F,d) of (F_i,d_i)'s is the unique forest of $\mathbb{F}$ such that its image by m is $m((F,d))$.

So m is injective on $\mathbb{F}$. By triangularity, m is injective on $\mathbb{F}_{\mathbf{H}_{hpo}}$ and we deduce that the restriction of Φ to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras. $\square$

2.2 Hopf algebras of contractions

2.2.1 Commutative case

In [CEFM11], D. Calaque, K. Ebrahimi-Fard and D. Manchon introduce a new coproduct, called in this paper the contraction coproduct, on the augmentation ideal of $\mathbf{H}_{CK}$ (see also [MS11]).

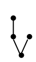
Definition 13 *Let F be a nonempty rooted forest and $\boldsymbol{e}$ a subset of $E(F)$. Then we denote by*

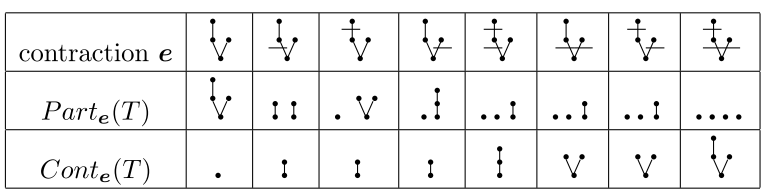
1. *$Part_{\boldsymbol{e}}(F)$ the subforest of F obtained by keeping all the vertices of F and the edges of $\boldsymbol{e}$,*
2. *$Cont_{\boldsymbol{e}}(F)$ the forest obtained by contracting each edge of $\boldsymbol{e}$ in F and identifying the two extremities of each edge of $\boldsymbol{e}$.*

We shall say that $\boldsymbol{e}$ is a contraction of F, $Part_{\boldsymbol{e}}(F)$ is the partition of F by $\boldsymbol{e}$ and $Cont_{\boldsymbol{e}}(F)$ is the contracted of F by $\boldsymbol{e}$. Each vertex of $Cont_{\boldsymbol{e}}(F)$ can be identified to a connected component of $Part_{\boldsymbol{e}}(F)$.

Remarks.
- If $\boldsymbol{e} = \emptyset$, then $Part_{\boldsymbol{e}}(F) = \underbrace{\bullet \dots \bullet}_{|F|_v \times}$ and $Cont_{\boldsymbol{e}}(F) = F$: this is the *empty contraction* of F.
- If $\boldsymbol{e} = E(F)$, then $Part_{\boldsymbol{e}}(F) = F$ and $Cont_{\boldsymbol{e}}(F) = \bullet$: this is the *total contraction* of F.

Notations. We shall write $\boldsymbol{e} \models E(F)$ if $\boldsymbol{e}$ is a contraction of F and $\boldsymbol{e} \mid\models E(F)$ if $\boldsymbol{e}$ is a nonempty, nontotal contraction of F.

Example. Let $T =$  be a rooted tree. Then



where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

Remarks. Let F be a nonempty rooted forest and $\boldsymbol{e} \models E(F)$.

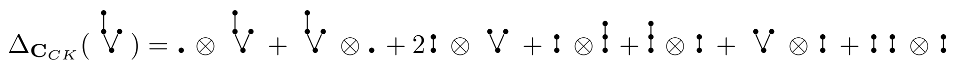
1. We have the following relation on the vertices degrees :
$$|F|_v = |Cont_{\boldsymbol{e}}(F)|_v + |Part_{\boldsymbol{e}}(F)|_v - l(Part_{\boldsymbol{e}}(F)).$$
2. Note $\overline{\boldsymbol{e}}$ the complementary to $\boldsymbol{e}$ in $E(F)$. Then $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$ and
$$|F|_e = |Cont_{\boldsymbol{e}}(F)|_e + |Part_{\boldsymbol{e}}(F)|_e . \tag{2.4}$$

Let $\mathbf{C}_{CK}$ be the quotient algebra $\mathbf{H}_{CK}/I_{CK}$ where I_{CK} is the ideal spanned by $\bullet - 1$. In others terms, one identifies the unit 1 (for the concatenation) with the tree $\bullet$. We note in the same way a rooted forest and his class in $\mathbf{C}_{CK}$. Then we define on $\mathbf{C}_{CK}$ a contraction coproduct on each forest $F \in \mathbf{C}_{CK}$:

$$\begin{aligned}
\Delta_{\mathbf{C}_{CK}}(F) &= \sum_{\boldsymbol{e} \models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F), \\
&= F \otimes \bullet + \bullet \otimes F + \sum_{\boldsymbol{e} \mid\models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F).
\end{aligned}$$

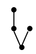
In particular, $\Delta_{\mathbf{C}_{CK}}(\bullet) = \bullet \otimes \bullet$.

Example.



We define an algebra morphism ε :

$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{C}_{CK} & \rightarrow & \mathbb{K} \\ F \text{ forest} & \mapsto & \delta_{F,\bullet}. \end{array} \right.$$

Then $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is a commutative Hopf algebra graded by the number of edges. $\mathbf{C}_{CK}$ is non cocommutative (see for example the coproduct of ).

Remark. We define inductively :

$$\Delta^{(0)}_{\mathbf{C}_{CK}} = Id, \qquad \Delta^{(1)}_{\mathbf{C}_{CK}} = \Delta_{\mathbf{C}_{CK}}, \qquad \Delta^{(k)}_{\mathbf{C}_{CK}} = (\Delta_{\mathbf{C}_{CK}} \otimes Id^{\otimes(k-1)}) \circ \Delta^{(k-1)}_{\mathbf{C}_{CK}}.$$

For all $k \in \mathbb{N}$, $\Delta^{(k)}_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \rightarrow \mathbf{C}_{CK}^{\otimes(k+1)}$. If F is a rooted forest with n edges, there are $(k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$:

- If $k = 0$, this is obvious.
- If $k > 0$, we have $\binom{n}{l}$ tensors $F^{(1)} \otimes F^{(2)}$ in $\Delta_{\mathbf{C}_{CK}}(F)$ such that the left term $F^{(1)}$ have l edges. By the induction hypothesis, there are k^l terms in $\Delta^{(k-1)}_{\mathbf{C}_{CK}}(F^{(1)})$. So there are $\sum_{0 \leq l \leq n} \binom{n}{l} k^l = (k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$.

We give the first numbers of trees $t_n^{\mathbf{C}_{CK}}$ and forests $f_n^{\mathbf{C}_{CK}}$:

n	0	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842
$f_n^{\mathbf{C}_{CK}}$	1	1	3	7	19	47	127	330	889	2378	6450

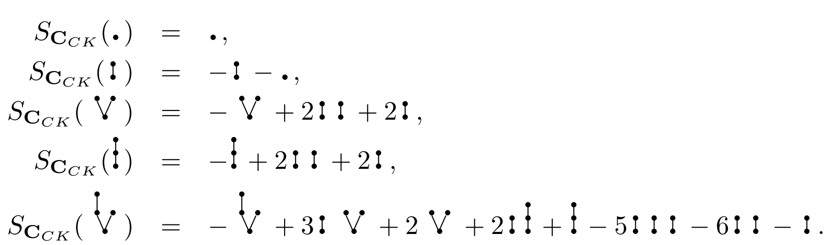
The first sequence is the sequence A000081 in [Slo].

We recall a combinatorial description of the antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \rightarrow \mathbf{C}_{CK}$ (see [CEFM11]) :

Proposition 14 *The antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \rightarrow \mathbf{C}_{CK}$ of the Hopf algebra $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is given (recursively with respect to number of edges) by the following formulas : for all forest $F \in \mathbf{C}_{CK}$,*

$$\begin{aligned} S_{\mathbf{C}_{CK}}(F) &= -F - \sum_{\boldsymbol{e} \| \models E(F)} S_{\mathbf{C}_{CK}}(Part_{\boldsymbol{e}}(F)) Cont_{\boldsymbol{e}}(F) \\ &= -F - \sum_{\boldsymbol{e} \| \models E(F)} Part_{\boldsymbol{e}}(F) S_{\mathbf{C}_{CK}}(Cont_{\boldsymbol{e}}(F)). \end{aligned}$$

Examples.

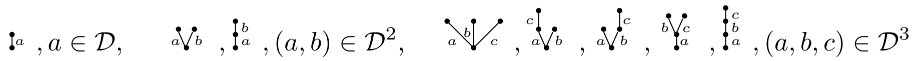


We now give a decorated version of $\mathbf{C}_{CK}$. Let $\mathcal{D}$ be a nonempty set. A rooted forest with their edges decorated by $\mathcal{D}$ is a couple (F, d) where F is a forest of $\mathbf{C}_{CK}$ and $d : E(F) \rightarrow \mathcal{D}$ is a map. We denote by $\mathbf{C}_{CK}^{\mathcal{D}}$ the $\mathbb{K}$-vector space spanned by rooted forests with edges decorated by $\mathcal{D}$.

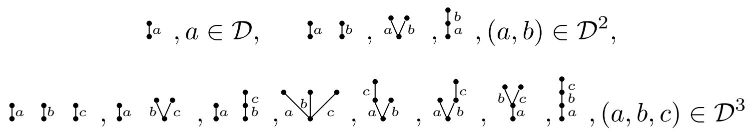
If (F,d) is a rooted forests with their edges decorated by $\mathcal{D}$ and if $v,w \in V(F)$ and $d \in \mathcal{D}$, we shall note $v \xrightarrow{d} w$ if there is an edge in F from v to w decorated by d.

Examples.

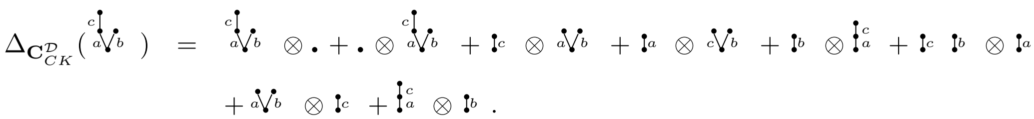
1. Rooted trees decorated by $\mathcal{D}$ with edges degree smaller than 3 :



2. Rooted forests decorated by $\mathcal{D}$ with edges degree smaller than 3 :



If $F \in \mathbf{C}_{CK}^{\mathcal{D}}$ and $\boldsymbol{e} \models E(F)$, then $Part_{\boldsymbol{e}}(F)$ and $Cont_{\boldsymbol{e}}(F)$ are naturally rooted forests with their edges decorated by $\mathcal{D}$: we keep the decoration of each edges. The vector space $\mathbf{C}_{CK}^{\mathcal{D}}$ is a Hopf algebra. Its product is given by the concatenation and its coproduct is the contraction coproduct. For example : if $(a,b,c) \in \mathcal{D}^3$,



Notation. The set of nonempty trees of $\mathbf{C}_{CK}$ (that is to say with at least one edge) will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}$. The set of nonempty trees with their edges decorated by $\mathcal{D}$ of $\mathbf{C}_{CK}^{\mathcal{D}}$ will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}^{\mathcal{D}}$.

2.2.2 Insertion operations

Let $\mathbf{T}_{CK}^{\mathcal{D}}$ be the $\mathbb{K}$-vector space having for basis $\mathbb{T}_{\mathbf{C}_{CK}}^{\mathcal{D}}$. In this section, we prove that $\mathbf{T}_{CK}^{\mathcal{D}}$ is equiped with two operations $\curlyvee$ and $\triangleright$ such that $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ is a commutative prelie algebra.

Definition 15 1. *A commutative prelie algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $\curlyvee, \triangleright : A \otimes A \to A$ such that $x \curlyvee y = y \curlyvee x$ for all $x,y \in A$ (that is to say $\curlyvee$ is commutative) and satisfying the following relations : for all $x,y,z \in A$,*

$$\begin{cases} (x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z), \\ x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z = y \triangleright (x \triangleright z) - (y \triangleright x) \triangleright z, \\ x \triangleright (y \curlyvee z) = (x \triangleright y) \curlyvee z + (x \triangleright z) \curlyvee y. \end{cases} \tag{2.5}$$

In other words, $(A, \curlyvee, \triangleright)$ is a commutative prelie algebra if $(A, \curlyvee)$ is a commutative algebra and $(A, \triangleright)$ is a left prelie algebra with a relationship between $\curlyvee$ and $\triangleright$.

2. *The commutative prelie operad, denoted $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$, is the operad such that $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebras are commutative prelie algebras.*

Remark. From this definition, it is clear that the operad $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$ is binary and quadratic (see [LV12] for a definition).

Notations.

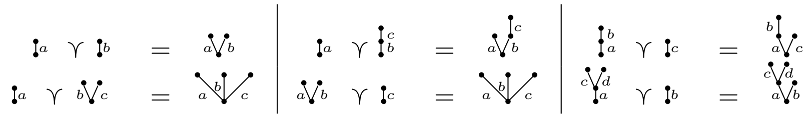
1. Let $T \in \mathbb{T}_{\mathbf{C}_{CK}}$ be a tree with at least one edge. We denote by $V^*(T) = V(T) \setminus \{R_T\}$ the set of vertices of T different from the root of T.
2. Let $T_1, T_2 \in \mathbb{T}_{\mathbf{C}_{CK}}$ and $v \in V(T_2)$. Then $T_1 \circ_v T_2$ is the tree obtained by identifying the root R_{T_1} of T_1 and the vertex v of T_2.

We define two $\mathbb{K}$-linear maps $\curlyvee : \mathbf{T}_{CK}^{\mathcal{D}} \otimes \mathbf{T}_{CK}^{\mathcal{D}} \to \mathbf{T}_{CK}^{\mathcal{D}}$ and $\triangleright : \mathbf{T}_{CK}^{\mathcal{D}} \otimes \mathbf{T}_{CK}^{\mathcal{D}} \to \mathbf{T}_{CK}^{\mathcal{D}}$ as follows : if $T_1, T_2 \in \mathbb{T}_{\mathbf{C}_{CK}}^{\mathcal{D}}$,

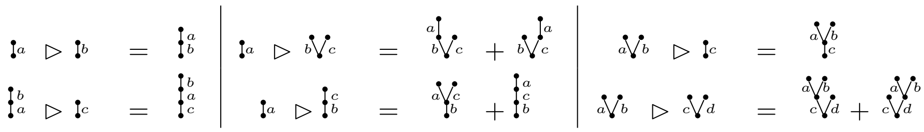
$$\begin{aligned} T_1 \curlyvee T_2 &= T_1 \circ_{R_{T_2}} T_2, \\ T_1 \triangleright T_2 &= \sum_{s \in V^*(T_2)} T_1 \circ_s T_2. \end{aligned}$$

Examples.

1. For the map $\curlyvee : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



2. For the map $\triangleright : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



Proposition 16 $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \triangleright)$ *is a $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra.*

Proof. Let $T_1, T_2, T_3 \in \mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$. Then

$$T_1 \curlyvee T_2 = T_1 \circ_{R_{T_2}} T_2 = T_2 \circ_{R_{T_1}} T_1 = T_2 \curlyvee T_1.$$

Moreover,

$$(T_1 \curlyvee T_2) \curlyvee T_3 = (T_1 \circ_{R_{T_2}} T_2) \circ_{R_{T_3}} T_3 = T_1 \circ_{R_{(T_2 \circ_{R_{T_3}} T_3)}} (T_2 \circ_{R_{T_3}} T_3) = T_1 \curlyvee (T_2 \curlyvee T_3).$$

Therefore $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee)$ is a commutative algebra.

$$\begin{aligned}
T_1 \triangleright (T_2 \triangleright T_3) &= \sum_{\substack{v \in V^*(T_3) \\ w \in V^*(T_2) \cup V^*(T_3)}} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} T_1 \circ_w (T_2 \circ_v T_3) + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} (T_1 \circ_w T_2) \circ_v T_3 + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= (T_1 \triangleright T_2) \triangleright T_3 + \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3).
\end{aligned}$$

So

$$\begin{aligned}
T_1 \triangleright (T_2 \triangleright T_3) - (T_1 \triangleright T_2) \triangleright T_3 &= \sum_{v, w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v, w \in V^*(T_3)} T_2 \circ_v (T_1 \circ_w T_3) \\
&= T_2 \triangleright (T_1 \triangleright T_3) - (T_2 \triangleright T_1) \triangleright T_3.
\end{aligned}$$

Therefore, $(\mathbf{T}^{\mathcal{D}}_{CK}, \triangleright)$ is a left prelie algebra.

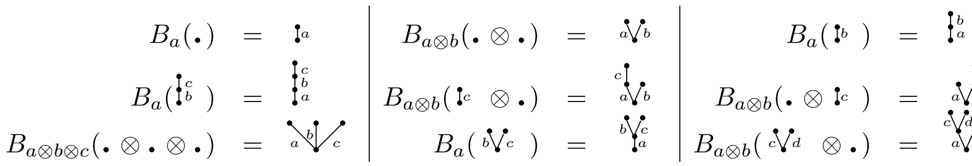
It remains to prove the last relation of (2.5) :

$$\begin{aligned}
T_1 \triangleright (T_2 \curlyvee T_3) &= \sum_{v \in V^*(T_2 \circ_{R_{T_3}} T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) \\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) \\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_3 \circ_{R_{T_2}} T_2) \\
&= \left(\sum_{v \in V^*(T_2)} T_1 \circ_v T_2 \right) \circ_{R_{T_3}} T_3 + \left(\sum_{v \in V^*(T_3)} T_1 \circ_v T_3 \right) \circ_{R_{T_2}} T_2 \\
&= (T_1 \triangleright T_2) \curlyvee T_3 + (T_1 \triangleright T_3) \curlyvee T_2.
\end{aligned}$$

$\square$

Theorem 17 *$(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ is generated as $\mathcal{ComPreLie}$-algebra by ℓ_d, $d \in \mathcal{D}$.*

Notation. To prove the previous proposition, we introduce a notation. Let $T_1, \ldots, T_k$ are trees (possibly empty) of $\mathbf{C}_{CK}^{\mathcal{D}}$ and $d_1, \ldots, d_k \in \mathcal{D}$. Then $B_{d_1 \otimes \ldots \otimes d_k}(T_1 \otimes \ldots \otimes T_k)$ is the tree obtained by grafting each T_i on a common root with an edge decorated by d_i. For examples, if $a, b, c, d \in \mathcal{D}$,

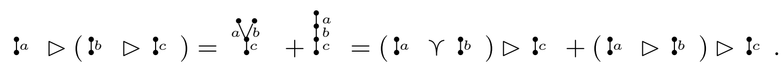


Proof. Let us prove that $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ is generated as $\mathcal{ComPreLie}$-algebra by ℓ_d, $d \in \mathcal{D}$ by induction on the edges degree n. If $n = 1$, this is obvious. Let $T \in \mathbf{T}_{CK}^{\mathcal{D}}$ be a tree of edges degree $n \geq 2$. Let k be an integer such that $T = B_{d_1 \otimes \ldots \otimes d_k}(T_1 \otimes \ldots \otimes T_k)$ with $d_1, \ldots, d_k \in \mathcal{D}$ and $T_1, \ldots, T_k$ trees (possibly empty) of $\mathbf{C}_{CK}^{\mathcal{D}}$. Then :

1. If $k = 1$, $T = B_{d_1}(T_1)$ with $|T_1|_e = n - 1 \geq 1$. By induction hypothesis, T_1 can be constructed from trees ℓ_d, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$. So $T = T_1 \triangleright \ell_{d_1}$ can be also constructed from trees ℓ_d, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$.
2. Suppose that $k \geq 2$. Then, for all i, $1 \leq |B_{d_i}(T_i)|_e \leq n - 1$. By induction hypothesis, the trees $B_{d_i}(T_i)$ can be constructed from trees ℓ_d, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$. So $T = B_{d_1}(T_1) \curlyvee \ldots \curlyvee B_{d_k}(T_k)$ can be also constructed from trees ℓ_d, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\triangleright$.

We conclude with the induction principle. $\square$

Remarks. $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \triangleright)$ is not the free $\mathcal{ComPreLie}$-algebra generated by ℓ_d, $d \in \mathcal{D}$. For example,



2.2.3 Noncommutative case

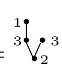
We give a noncommutative version of $\mathbf{C}_{CK}$. To do this, we work on the algebra $\mathbf{H}_{po}$.

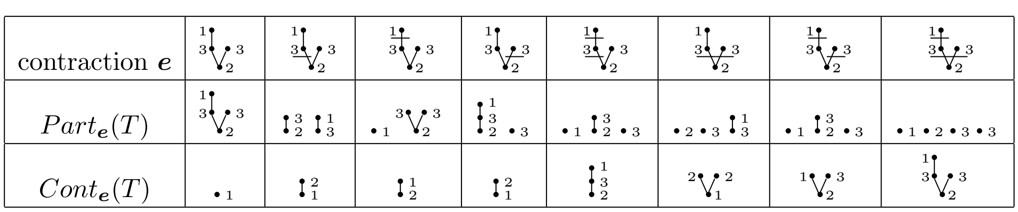
Definition 18 *Let (F, σ^F) be a nonempty preordered forest. In particular, F is a nonempty rooted forest. Let $\boldsymbol{e}$ be a contraction of F, $Part_{\boldsymbol{e}}(F)$ the partition of F by $\boldsymbol{e}$ and $Cont_{\boldsymbol{e}}(F)$ the contracted of F by $\boldsymbol{e}$ (see definition 13). Then :*

1. *$Part_{\boldsymbol{e}}(F)$ is a preordered forest $(Part_{\boldsymbol{e}}(F), \sigma^P)$ where $\sigma^P : v \in V(Part_{\boldsymbol{e}}(F)) \mapsto \sigma^F(v)$. In other words, we keep the initial preorder of the vertices of F in $Part_{\boldsymbol{e}}(F)$.*
2. *$Cont_{\boldsymbol{e}}(F)$ is also a preordered forest $(Cont_{\boldsymbol{e}}(F), \sigma^C)$ where $\sigma^C : V(Cont_{\boldsymbol{e}}(F)) \to \{1, \ldots, p\}$ is the surjection ($p \leq |Cont_{\boldsymbol{e}}(F)|_v$) such that if A, B are two connected components of $Part_{\boldsymbol{e}}(F)$, if a (resp. b) is the vertex obtained by contracting A (resp. B) in F, then*

$$\begin{cases} \sigma^F(R_A) < \sigma^F(R_B) & \Longrightarrow & \sigma^C(a) < \sigma^C(b), \\ \sigma^F(R_A) = \sigma^F(R_B) & \Longrightarrow & \sigma^C(a) = \sigma^C(b), \\ \sigma^F(R_A) > \sigma^F(R_B) & \Longrightarrow & \sigma^C(a) > \sigma^C(b). \end{cases} \tag{2.6}$$

In other words, we contract each connected component of $Part_{\boldsymbol{e}}(F)$ to its root and we keep the initial preorder of the roots.

Example. Let $T =$  be a preordered tree. Then



where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

Let I_{po} be the ideal of $\mathbf{H}_{po}$ generated by the elements $F\bullet_i - \tilde{F}$ with $F\bullet_i \in \mathbf{C}'_{po}$ and $\tilde{F}$ the forest contructed from $F\bullet_i$ by deleting the vertex $\bullet_i$ and keeping the same preorder on $V(F)$. For example,

– if $F_{\bullet_i} = {}^{1}\mathbf{V}^{3}_{3}\ \bullet_2$ then $\tilde{F} = {}^{1}\mathbf{V}^{2}_{2}$,

– if $F_{\bullet_i} = {}^{\overset{2}{1}}\mathbf{V}^{3}_{2}\ \bullet_2$ then $\tilde{F} = {}^{\overset{2}{1}}\mathbf{V}^{3}_{2}$.

Let $\mathbf{C}_{po}$ be the quotient algebra $\mathbf{H}_{po}/I_{po}$. So one identifies the unit 1 (for the concatenation) with the tree $\bullet_1$. Note that $\mathbf{C}_{po}$ is a graded algebra by the number of edges. We note in the same way a forest and his class in $\mathbf{C}_{po}$. We define on $\mathbf{C}_{po}$ a contraction coproduct on each preordered forest $F \in \mathbf{C}_{po}$:

$$\begin{aligned}\Delta_{\mathbf{C}_{po}}(F) &= \sum_{\boldsymbol{e}\models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F),\\ &= F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_{\boldsymbol{e}\Vert\!\models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F).\end{aligned}$$

Examples.

$$\begin{aligned}
\Delta_{\mathbf{C}_{po}}(\bullet_1) &= \bullet_1 \otimes \bullet_1\\
\Delta_{\mathbf{C}_{po}}(\mathbf{I}^{1}_{2}) &= \mathbf{I}^{1}_{2} \otimes \bullet_1 + \bullet_1 \otimes \mathbf{I}^{1}_{2}\\
\Delta_{\mathbf{C}_{po}}({}^{1}\mathbf{V}^{2}_{2}) &= {}^{1}\mathbf{V}^{2}_{2} \otimes \bullet_1 + \bullet_1 \otimes {}^{1}\mathbf{V}^{2}_{2} + \mathbf{I}^{1}_{2} \otimes \mathbf{I}^{1}_{1} + \mathbf{I}^{1}_{1} \otimes \mathbf{I}^{1}_{2}\\
\Delta_{\mathbf{C}_{po}}(\mathbf{I}^{4}_{2}\mathbf{I}^{1}_{3}) &= \mathbf{I}^{4}_{2}\mathbf{I}^{1}_{3} \otimes \bullet_1 + \bullet_1 \otimes \mathbf{I}^{4}_{2}\mathbf{I}^{1}_{3} + \mathbf{I}^{2}_{1} \otimes \mathbf{I}^{1}_{2} + \mathbf{I}^{1}_{2} \otimes \mathbf{I}^{2}_{1}\\
\Delta_{\mathbf{C}_{po}}({}^{\overset{1}{3}}\mathbf{V}^{3}_{2}) &= {}^{\overset{1}{3}}\mathbf{V}^{3}_{2} \otimes \bullet_1 + \bullet_1 \otimes {}^{\overset{1}{3}}\mathbf{V}^{3}_{2} + \mathbf{I}^{3}_{2}\mathbf{I}^{1}_{3} \otimes \mathbf{I}^{2}_{1} + {}^{2}\mathbf{V}^{2}_{1} \otimes \mathbf{I}^{1}_{2} + \mathbf{I}^{1}_{\substack{3\\2}} \otimes \mathbf{I}^{2}_{1} + \mathbf{I}^{2}_{1} \otimes \mathbf{I}^{1}_{\substack{3\\2}}\\
&\quad + \mathbf{I}^{1}_{2} \otimes {}^{2}\mathbf{V}^{2}_{1} + \mathbf{I}^{2}_{1} \otimes {}^{1}\mathbf{V}^{3}_{2}\\
\Delta_{\mathbf{C}_{po}}({}^{2}\mathbf{V}^{5}_{3}\mathbf{I}^{1}_{4}) &= {}^{2}\mathbf{V}^{5}_{3}\mathbf{I}^{1}_{4} \otimes \bullet_1 + \bullet_1 \otimes {}^{2}\mathbf{V}^{5}_{3}\mathbf{I}^{1}_{4} + \mathbf{I}^{1}_{2} \otimes \mathbf{I}^{4}_{2}\mathbf{I}^{1}_{3} + \mathbf{I}^{2}_{1} \otimes \mathbf{I}^{2}_{3}\mathbf{I}^{1}_{4} + \mathbf{I}^{1}_{2} \otimes {}^{1}\mathbf{V}^{3}_{2}\\
&\quad + \mathbf{I}^{2}_{3}\mathbf{I}^{1}_{4} \otimes \mathbf{I}^{2}_{1} + \mathbf{I}^{4}_{2}\mathbf{I}^{1}_{3} \otimes \mathbf{I}^{1}_{2} + {}^{1}\mathbf{V}^{3}_{2} \otimes \mathbf{I}^{1}_{2}
\end{aligned}$$

Remark. $\Delta_{\mathbf{C}_{po}}$ is non cocommutative (see for example the coproduct of ${}^{\overset{1}{3}}\mathbf{V}^{3}_{2}$). In particular, if T is a preordered tree and $\boldsymbol{e} \models E(T)$, $Cont_{\boldsymbol{e}}(T)$ is a preordered tree and $Part_{\boldsymbol{e}}(T)$ can be disconnected. The second component of the coproduct is linear : a tree instead of a polynomial in trees. This is a right combinatorial Hopf algebra (see [LR10]).

Proposition 19 *1. $\Delta_{\mathbf{C}_{po}}$ is a graded algebra morphism.*

2. $\Delta_{\mathbf{C}_{po}}$ is coassociative.

Proof.

1. Let F, G be two preordered forests. Then

$$\begin{aligned}\Delta_{\mathbf{C}_{po}}(FG) &= \sum_{\boldsymbol{e}\models E(FG)} Part_{\boldsymbol{e}}(FG) \otimes Cont_{\boldsymbol{e}}(FG)\\
&= \sum_{\boldsymbol{e}\models E(F), \boldsymbol{f}\models E(G)} (Part_{\boldsymbol{e}}(F)Part_{\boldsymbol{f}}(G)) \otimes (Cont_{\boldsymbol{e}}(F)Cont_{\boldsymbol{f}}(G))\\
&= \left(\sum_{\boldsymbol{e}\models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F)\right)\left(\sum_{\boldsymbol{f}\models E(G)} Part_{\boldsymbol{f}}(G) \otimes Cont_{\boldsymbol{f}}(G)\right)\\
&= \Delta_{\mathbf{C}_{po}}(F)\Delta_{\mathbf{C}_{po}}(G),\end{aligned}$$

and $\Delta_{\mathbf{C}_{po}}$ is an algebra morphism. It is a graded algebra morphism with (2.4).

2. Let F be a nonempty preordered forest. Then

$$\begin{aligned}&(\Delta_{\mathbf{C}_{po}} \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F)\\
&= \sum_{\boldsymbol{e}\models E(F)} \Delta_{\mathbf{C}_{po}}(Part_{\boldsymbol{e}}(F)) \otimes Cont_{\boldsymbol{e}}(F)\\
&= \sum_{\boldsymbol{e}\models E(F)}\ \sum_{\boldsymbol{f}\models E(Part_{\boldsymbol{e}}(F))} Part_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F)) \otimes Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F)) \otimes Cont_{\boldsymbol{e}}(F)\\
&= \sum_{\boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)} Part_{\boldsymbol{f}}(F) \otimes Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F)) \otimes Cont_{\boldsymbol{e}}(F),\end{aligned}$$

and

$$\begin{aligned}
& (Id \otimes \Delta_{\mathbf{C}_{po}}) \circ \Delta_{\mathbf{C}_{po}}(F) \\
= & \sum_{\boldsymbol{f} \models E(F)} Part_{\boldsymbol{f}}(F) \otimes \Delta_{\mathbf{C}_{po}}(Cont_{\boldsymbol{f}}(F)) \\
= & \sum_{\boldsymbol{f} \models E(F)} \sum_{\boldsymbol{e} \models E(Cont_{\boldsymbol{f}}(F))} Part_{\boldsymbol{f}}(F) \otimes Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F)) \otimes Cont_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F)) \\
= & \sum_{\boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}} Part_{\boldsymbol{f}}(F) \otimes Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F)) \otimes Cont_{\boldsymbol{e} \cup \boldsymbol{f}}(F),
\end{aligned}$$

where to the last equality we use that $E(Cont_{\boldsymbol{f}}(F)) = \overline{\boldsymbol{f}}$ the complement of $\boldsymbol{f}$ in $E(F)$ and $Cont_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F)) = Cont_{\boldsymbol{e} \cup \boldsymbol{f}}(F)$.

Remark that $\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)\}$ and $\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}\}$ are in bijection :

$$\left\{ \begin{array}{rcl}
\{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)\} & \rightarrow & \{(\boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}\} \\
(\boldsymbol{e}, \boldsymbol{f}) & \rightarrow & (\boldsymbol{e} \setminus \boldsymbol{f}, \boldsymbol{f}) \\
(\boldsymbol{e} \cup \boldsymbol{f}, \boldsymbol{f}) & \leftarrow & (\boldsymbol{e}, \boldsymbol{f}).
\end{array} \right.$$

Moreover,
- in $Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))$ with $\boldsymbol{f} \subseteq \boldsymbol{e} \subseteq E(F)$: the edges belong to $\boldsymbol{e} \cap \overline{\boldsymbol{f}} = \boldsymbol{e} \setminus \boldsymbol{f}$; the vertices are the connected components of $Part_{\boldsymbol{e} \cap \boldsymbol{f}}(F) = Part_{\boldsymbol{f}}(F)$. The preorder on the vertices is given by the preorder on the roots of the connected components of $Part_{\boldsymbol{f}}(F)$.
- in $Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))$ with $\boldsymbol{f} \models E(F), \boldsymbol{e} \subseteq \overline{\boldsymbol{f}}$: the edges belong to $\overline{\boldsymbol{f}} \cap \boldsymbol{e} = \boldsymbol{e} \setminus \boldsymbol{f} = \boldsymbol{e}$; the vertices are the connected components of $Part_{\boldsymbol{f}}(F)$. As in the precedent case, the preorder on the vertices is given by the preorder on the roots of the connected components of $Part_{\boldsymbol{f}}(F)$.

So $Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))$ and $Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))$ are the same forests with the same preorder on the vertices.

Therefore $(\Delta_{\mathbf{C}_{po}} \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) = (Id \otimes \Delta_{\mathbf{C}_{po}}) \circ \Delta_{\mathbf{C}_{po}}(F)$.

□

We now define

$$\varepsilon : \left\{ \begin{array}{rcl}
\mathbf{C}_{po} & \rightarrow & \mathbb{K} \\
F \text{ forest} & \mapsto & \delta_{F, \bullet_1}.
\end{array} \right.$$

ε is an algebra morphism.

Proposition 20 *ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$.*

Proof. Let F be a forest $\in \mathbf{C}_{po}$. We use the Sweedler notation :

$$\Delta_{\mathbf{C}_{po}}(F) = F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}.$$

Then

$$\begin{aligned}
(\varepsilon \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) &= \varepsilon(F) \bullet_1 + \varepsilon(\bullet_1) F + \sum_F \varepsilon(F^{(1)}) \otimes F^{(2)} = F, \\
(Id \otimes \varepsilon) \circ \Delta_{\mathbf{C}_{po}}(F) &= F \varepsilon(\bullet_1) + \bullet_1 \varepsilon(F) + \sum_F F^{(1)} \varepsilon(F^{(2)}) = F.
\end{aligned}$$

Therefore ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$. □

As $(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is graded (by the number of edges) and connected, we have the following theorem :

Theorem 21 *$(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is a Hopf algebra.*

We denote the antipode of the Hopf algebra $\mathbf{C}_{po}$ by $S_{\mathbf{C}_{po}}$. We have the same combinatorial description of $S_{\mathbf{C}_{po}}$ as the commutative case (see proposition 14). We give some values of $S_{\mathbf{C}_{po}}$:

- In edges degree 0, $S_{\mathbf{C}_{po}}(\bullet_1) = \bullet_1$.
- In edges degree 1, $S_{\mathbf{C}_{po}}(\mathrm{I}^1_1) = -\mathrm{I}^1_1 - \bullet_1$, $S_{\mathbf{C}_{po}}(\mathrm{I}^2_1) = -\mathrm{I}^2_1 - \bullet_1$ and $S_{\mathbf{C}_{po}}(\mathrm{I}^1_2) = -\mathrm{I}^1_2 - \bullet_1$.
- In edges degree 2,

$$\begin{array}{rcl}
S_{\mathbf{C}_{po}}({}^2\mathrm{V}^2_1) & = & -{}^2\mathrm{V}^2_1 + 2\mathrm{I}^2_1\mathrm{I}^4_3 + 2\mathrm{I}^2_1,\\
S_{\mathbf{C}_{po}}({}^1\mathrm{V}^3_2) & = & -{}^1\mathrm{V}^3_2 + \mathrm{I}^1_2\mathrm{I}^4_3 + \mathrm{I}^2_1\mathrm{I}^3_4 + \mathrm{I}^2_1 + \mathrm{I}^1_2,\\
S_{\mathbf{C}_{po}}(\mathrm{I}^{\substack{1\\2}}_{1}) & = & -\mathrm{I}^{\substack{1\\2}}_{1} + \mathrm{I}^2_1\mathrm{I}^3_1 + \mathrm{I}^1_2\mathrm{I}^4_3 + \mathrm{I}^1_1 + \mathrm{I}^2_1,\\
S_{\mathbf{C}_{po}}(\mathrm{I}^3_2\mathrm{I}^1_3) & = & -\mathrm{I}^3_2\mathrm{I}^1_3 + \mathrm{I}^2_1\mathrm{I}^3_4 + \mathrm{I}^1_2\mathrm{I}^4_3 + \mathrm{I}^1_2 + \mathrm{I}^2_1.
\end{array}$$

- In edges degree 3,

$$\begin{array}{rcl}
S_{\mathbf{C}_{po}}({}^{\substack{1\\3}}\mathrm{V}^3_2) & = & -{}^{\substack{1\\3}}\mathrm{V}^3_2 + \mathrm{I}^3_2\mathrm{I}^1_3\mathrm{I}^5_4 - \mathrm{I}^2_1\mathrm{I}^3_4\mathrm{I}^6_5 - \mathrm{I}^1_2\mathrm{I}^4_3\mathrm{I}^6_5 - \mathrm{I}^1_2\mathrm{I}^4_3 - \mathrm{I}^2_1\mathrm{I}^4_3 + {}^2\mathrm{V}^2_1\mathrm{I}^3_4\\
& & -2\mathrm{I}^2_1\mathrm{I}^4_3\mathrm{I}^5_6 - 2\mathrm{I}^2_1\mathrm{I}^3_4 + \mathrm{I}^{\substack{1\\3}}_{2}\mathrm{I}^4_3 - \mathrm{I}^1_2\mathrm{I}^4_3\mathrm{I}^6_5 - \mathrm{I}^2_1\mathrm{I}^4_3 - \mathrm{I}^2_1\mathrm{I}^3_4\mathrm{I}^6_5 - \mathrm{I}^1_2\mathrm{I}^4_3\\
& & +\mathrm{I}^2_1\mathrm{I}^{\substack{3\\5}}_{4} + \mathrm{I}^{\substack{1\\3}}_{2} + \mathrm{I}^1_2\,{}^4\mathrm{V}^4_3 + {}^2\mathrm{V}^2_1 + \mathrm{I}^2_1\,{}^3\mathrm{V}^5_4 + {}^1\mathrm{V}^3_2.
\end{array}$$

Let $\mathbf{C}'_{hpo}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-preordered forests, $\mathbf{C}'_o$ be the $\mathbb{K}$-algebra spanned by nonempty ordered forests, $\mathbf{C}'_{ho}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-ordered forests and $\mathbf{C}'_{NCK}$ be the $\mathbb{K}$-algebra spanned by nonempty planar forests. We consider the quotients $\mathbf{C}_{hpo} = \mathbf{C}'_{hpo}/(I_{po}\cap\mathbf{C}'_{hpo})$, $\mathbf{C}_o = \mathbf{C}'_o/(I_{po}\cap\mathbf{C}'_o)$, $\mathbf{C}_{ho} = \mathbf{C}'_{ho}/(I_{po}\cap\mathbf{C}'_{ho})$ and $\mathbf{C}_{NCK} = \mathbf{C}'_{NCK}/(I_{po}\cap\mathbf{C}'_{NCK})$. We have in this case a similar diagram to (2.2) :

$$\begin{tikzcd}
\mathbf{C}_{NCK} \arrow[r, hook] & \mathbf{C}_{ho} \arrow[r, hook] \arrow[d, hook] & \mathbf{C}_o \arrow[d, hook] \\
 & \mathbf{C}_{hpo} \arrow[r, hook] & \mathbf{C}_{po}
\end{tikzcd}$$

where the arrows $\hookrightarrow$ are injective morphisms of algebras. But they are not always morphisms of Hopf algebras (for the contraction coproduct) :

Theorem 22 *1. $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.*

2. $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.

3. $\mathbf{C}_{ho}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_o$ and of the Hopf algebra $\mathbf{C}_{hpo}$.

4. $\mathbf{C}_{NCK}$ is a left comodule of the Hopf algebra $\mathbf{C}_{ho}$.

Notations. We denote by $\Delta_{\mathbf{C}_{hpo}}, \Delta_{\mathbf{C}_o}, \Delta_{\mathbf{C}_{ho}}$ the restrictions to $\Delta_{\mathbf{C}_{po}}$ on $\mathbf{C}_{hpo}, \mathbf{C}_o, \mathbf{C}_{ho}$.

Remark. $\mathbf{C}_{NCK}$ is not a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{ho}$. For example, ${}^{\substack{3\\2}}\mathrm{V}^4_1 \in \mathbf{C}_{NCK}$ and

$$\begin{array}{rcl}
\Delta_{\mathbf{C}_{ho}}({}^{\substack{3\\2}}\mathrm{V}^4_1) & = & {}^{\substack{3\\2}}\mathrm{V}^4_1 \otimes \bullet_1 + \bullet_1 \otimes {}^{\substack{3\\2}}\mathrm{V}^4_1 + \mathrm{I}^{\substack{3\\2}}_{1} \otimes \mathrm{I}^2_1 + {}^2\mathrm{V}^3_1 \otimes \mathrm{I}^2_1 + 2\mathrm{I}^2_1 \otimes {}^2\mathrm{V}^3_1\\
& & +\mathrm{I}^2_1 \otimes \mathrm{I}^{\substack{3\\2}}_{1} + \mathrm{I}^4_1\mathrm{I}^3_2 \otimes \mathrm{I}^2_1.
\end{array}$$

Then $\mathrm{I}^4_1\mathrm{I}^3_2 \otimes \mathrm{I}^2_1 \notin \mathbf{C}_{NCK}\otimes\mathbf{C}_{NCK}$.

Proof.

1. $\mathbf{C}_{hpo}$ is a subalgebra of $\mathbf{C}_{po}$. Let us prove that if $(F,\sigma^F) \in \mathbf{C}_{hpo}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F),\sigma^C)$ and $(Part_{\boldsymbol{e}}(F),\sigma^P) \in \mathbf{C}_{hpo}$.
   If $a,b \in V(Part_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$ then a,b are the vertices of a subtree of $(F,\sigma^F) \in \mathbf{C}_{hpo}$ and $\sigma^F(a) > \sigma^F(b)$. With definition 18, $\sigma^P(a) > \sigma^P(b)$. So $(Part_{\boldsymbol{e}}(F),\sigma^P) \in \mathbf{C}_{hpo}$.
   If $a,b \in V(Cont_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$, then a and b are the vertices obtained by contracting two connected components A and B of $Part_{\boldsymbol{e}}(F)$. As $a \twoheadrightarrow b$, $R_A \twoheadrightarrow R_B$ and as $(F,\sigma^F) \in \mathbf{C}_{hpo}$, $\sigma^F(R_A) > \sigma^F(R_B)$. Then, by definition 18, $\sigma^C(a) > \sigma^C(b)$. So $(Cont_{\boldsymbol{e}}(F),\sigma^C) \in \mathbf{C}_{hpo}$.
   Therefore if $(F,\sigma^F) \in \mathbf{C}_{hpo}$, $\Delta_{\mathbf{C}_{po}}(F) \in \mathbf{C}_{hpo}\otimes\mathbf{C}_{hpo}$ and $\mathbf{C}_{hpo}$ is a Hopf subalgebra of $\mathbf{C}_{po}$.

2. $\mathbf{C}_o$ is a subalgebra of $\mathbf{C}_{po}$. Let $(F,\sigma^F) \in \mathbf{C}_o$ and $\boldsymbol{e} \models E(F)$. Let us show that $(Cont_{\boldsymbol{e}}(F),\sigma^C)$ and $(Part_{\boldsymbol{e}}(F),\sigma^P) \in \mathbf{C}_o$, that is to say that σ^C and σ^P are bijective.
   By definition 18, σ^P is bijective because we keep the initial order of the vertices of F in $Part_{\boldsymbol{e}}(F)$. By definition, σ^C is a surjection. Let $a, b \in V(Cont_{\boldsymbol{e}}(F))$ such that $\sigma^C(a) = \sigma^C(b)$ and A and B be the two connected components of $Part_{\boldsymbol{e}}(F)$ associated with a and b. With (2.6), $\sigma^F(R_A) = \sigma^F(R_B)$ and $R_A = R_B$ because σ^F is bijective. So $A = B$, $a = b$ and σ^C is injective.
   Therefore σ^C and σ^P are bijective and $\mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{ho}$.
3. As $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$ and $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$, $\mathbf{C}_{ho} = \mathbf{C}_{hpo} \cap \mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{hpo}$ and $\mathbf{C}_o$.
4. Let us prove that if $(F,\sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F),\sigma^C) \in \mathbf{C}_{NCK}$. As $\mathbf{C}_{ho}$ is a Hopf algebra, $(Cont_{\boldsymbol{e}}(F),\sigma^C) \in \mathbf{C}_{ho}$. So, if $a, b \in V(Cont_{\boldsymbol{e}}(F))$, such that $a \twoheadrightarrow b$ then $\sigma^C(a) \geq \sigma^C(b)$.
   Moreover, if $a, b, c \in V(Cont_{\boldsymbol{e}}(F))$ three distinct vertices such that $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b. The vertices a, b and c are obtained by contracting of connected components A, B and C in F. As $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b, $R_A \twoheadrightarrow R_C$, $R_B \twoheadrightarrow R_C$ and R_A is on the left of R_B. As $(F,\sigma^F) \in \mathbf{C}_{NCK}$, $\sigma^F(R_A) < \sigma^F(R_B)$. So $\sigma^C(a) < \sigma^C(b)$.
   Therefore if $(F,\sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F),\sigma^C) \in \mathbf{C}_{NCK}$. Consequently, $\Delta_{\mathbf{C}_{ho}}(\mathbf{C}_{NCK}) \subseteq \mathbf{C}_{ho} \otimes \mathbf{C}_{NCK}$.

□

2.2.4 Formal series

The algebras $\mathbf{C}_{po}$, $\mathbf{C}_{hpo}$, $\mathbf{C}_o$, $\mathbf{C}_{ho}$ and $\mathbf{C}_{NCK}$ are graded by the number of edges.

In the ordered case, we give some values in a small degrees :

n	1	2	3	4	5	6	7	8
$f_n^{\mathbf{C}_o}$	2	9	76	805	10626	167839	3091768	65127465

These is the sequence A105785 in [Slo].

Let us now study the heap-ordered case. We denote by $f_{n,l}^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edges degree n and of length l, and by $f_n^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edges degree n. In small degree, we have the following values :

$$\begin{cases} f_{0,0}^{\mathbf{C}_{ho}} = f_{1,1}^{\mathbf{C}_{ho}} = 1, & \\ f_{0,l}^{\mathbf{C}_{ho}} = 0 & \text{for all } l \geq 1, \\ f_{1,l}^{\mathbf{C}_{ho}} = 0 & \text{for all } l \neq 1, \\ f_{n,0}^{\mathbf{C}_{ho}} = 0 & \text{for all } n \neq 1. \end{cases}$$

Let n and l be two integers ≥ 1. To obtain a forest $F \in \mathbf{C}_{ho}$ of edges degree n and of length l (so $|F|_v = n + l$), we have two cases :

1. We consider a forest $G \in \mathbf{C}_{ho}$ of edges degree $n-1$ and of length l and we graft the vertex $n+l$ on the vertex i of G. For each forest G, we have $n+l-1$ possibilities.
2. We consider a forest $G \in \mathbf{C}_{ho}$ of edges degree $n-1$ and of length $l-1$. Then, for all $i \in \{1, \ldots, n+l-1\}$, the forest $\tilde{G} \updownarrow_i^{n+l}$ of edges degree n and of length l is an element of $\mathbf{C}_{ho}$ (where $\tilde{G}$ is the same forest than G where, for all $j \geq i$, the vertex j in G is the vertex $j+1$ in $\tilde{G}$). For each forest G, we have $n+l-1$ possibilities.

So

$$f_{n,l}^{\mathbf{C}_{ho}} = (n+l-1) f_{n-1,l}^{\mathbf{C}_{ho}} + (n+l-1) f_{n-1,l-1}^{\mathbf{C}_{ho}}.$$

We give some values of $f_{n,l}^{\mathbf{C}_{ho}}$ in a small degrees and in a small lengths :

n \ l	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	3	0	0	0
3	0	6	20	15	0	0
4	0	24	130	210	105	0
5	0	120	924	2380	2520	945

Note that $f_{n,1}^{\mathbf{C}_{ho}} = n!$ for all $n \geq 1$. With the formula $f_n^{\mathbf{C}_{ho}} = \sum_{l\geq 0} f_{n,l}^{\mathbf{C}_{ho}}$, we obtain the number of forests of edges degree n. This gives :

n	0	1	2	3	4	5	6
$f_n^{\mathbf{C}_{ho}}$	0	1	5	41	469	6889	123605

This is the sequence A032188 in [Slo].

Remark. Consider the map $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ defined by induction as follows. If $F = 1$, $\varphi(F) = 1$ and if $F = \bullet_1$, $\varphi(F) = (1)$. Let $F \in \mathbf{H}_{ho}$ be a forest of vertices degree n and v the vertex indexed by n. As F is a heap-ordered forest, two cases are possible :

- The vertex v is an isolated vertex. We denote by G the heap-ordered forest obtained by deleting the vertex v of F. Thus $\varphi(G) = \tau'$ is well defined by induction. Then $\varphi(F)$ is the permutation τ defined by

$$\left\{ \begin{array}{rcll} \tau(i) & = & \tau'(i) & \text{if } i \neq n \\ \tau(n) & = & n. & \end{array} \right.$$

- The vertex v is a leaf and we denote by k the index of v' with $v \to v'$. Similarly, we denote by G the heap-ordered forest obtained by deleting the vertex v of F. $\varphi(G) = \tau'$ is well defined by induction and $\varphi(F)$ is the permutation τ defined by

$$\left\{ \begin{array}{rcll} \tau(i) & = & \tau'(i) & \text{if } i \neq k \\ \tau(k) & = & n & \\ \tau(n) & = & \tau'(k). & \end{array} \right.$$

Then $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ is a bijective map. Remark that, if $F \in \mathbb{F}_{\mathbf{H}_{ho}}$, each connected component of F corresponds to one cycle in the writing of $\varphi(F)$ in product of disjoint cycles. Moreover, the restriction of φ to the forests of $\mathbf{C}_{ho}$ is a bijective map with values in the set of permutations without fixed point.

In the planar case, we can obtain the formal series. Let $t_n^{\mathbf{C}_{NCK}}$ be the number of trees of $\mathbf{C}_{NCK}$ of edges degree n and $f_n^{\mathbf{C}_{NCK}}$ be the number of forests of $\mathbf{C}_{NCK}$ of edges degree n. We put $T_{\mathbf{C}_{NCK}}(x) = \sum_{k\geq 0} t_k^{\mathbf{C}_{NCK}} x^k$ and $F_{\mathbf{C}_{NCK}}(x) = \sum_{k\geq 0} f_k^{\mathbf{C}_{NCK}} x^k$. Then :

Proposition 23 *The formal series $T_{\mathbf{C}_{NCK}}$ and $F_{\mathbf{C}_{NCK}}$ are given by :*

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1-2x-\sqrt{1-4x}}{2x}, \qquad F_{\mathbf{C}_{NCK}}(x) = \frac{2x}{4x-1+\sqrt{1-4x}}.$$

Proof. With formula (1.1), we deduce that :

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2x} - 1 = \frac{1-2x-\sqrt{1-4x}}{2x}.$$

$\mathbf{C}_{NCK}$ is freely generated by the trees, therefore

$$F_{\mathbf{C}_{NCK}}(x) = \frac{1}{1-T_{\mathbf{C}_{NCK}}(x)} = \frac{2x}{4x-1+\sqrt{1-4x}}$$

□

Then for all $n \geq 1$ $t_n^{\mathbf{C}_{NCK}} = \frac{1}{n+1}\binom{2n}{n}$ is the n-Catalan number, $f_n^{\mathbf{C}_{NCK}} = \binom{2n-1}{n}$ and this gives :

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{NCK}}$	1	2	5	14	42	132	429	1430	4862	16796
$f_n^{\mathbf{C}_{NCK}}$	1	3	10	35	126	462	1716	6435	24310	92378

These are the sequences A000108 and A088218 in [Slo].

2.3 Hopf algebra morphisms

Recall that the tensor algebra $T(V)$ over a $\mathbb{K}$-vector space V is the tensor module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots$$

equipped with the concatenation.

Dually, the tensor coalgebra $T^c(V)$ over a $\mathbb{K}$-vector space V is the tensor module (as above) equiped with the coassociative coproduct $\Delta_{\mathcal{A}ss}$ called deconcatenation :

$$\Delta_{\mathcal{A}ss}((v_1 \ldots v_n)) = \sum_{i=0}^{n} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n).$$

We will say that a graded bialgebra $\mathbf{H}$ is cofree if, as a graded coalgebra, it is isomorphic to $T^c(Prim(\mathbf{H}))$ (for more details, see [LR06]).

We give the useful following lemma :

Lemma 24 *Let (A, Δ, ε) be a graded cofree Hopf algebra. Then*

$$Ker(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta}) = Im(\tilde{\Delta}).$$

Proof. Indeed, if $x = \sum a_{w,w'} w \otimes w' \in Ker(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta})$,

$$\sum_{w_1 w_2 = w} a_{w,w'} w_1 \otimes w_2 \otimes w' = \sum_{w'_1 w'_2 = w'} a_{w,w'} w \otimes w'_1 \otimes w'_2.$$

So $a_{w_1 w_2, w_3} = a_{w_1, w_2 w_3}$ for all words w_1, w_2, w_3 different from the unit. We put $b_{ww'} = a_{w,w'}$. Then

$$x = \sum b_w \left(\sum_{w_1 w_2 = w} w_1 \otimes w_2 \right) = \tilde{\Delta}\left(\sum b_w w\right) \in Im(\tilde{\Delta}).$$

The coassociativity of $\tilde{\Delta}$ implies the other inclusion. □

2.3.1 From $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map.

Theorem 25 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}_{CK}^{\mathcal{D}} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}_{\mathbf{H}_{CK}}^{\mathcal{D}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{H}_{CK}^{\mathcal{D}} \arrow[r, "\Phi"] & \mathbf{Sh}^{\mathcal{D}} \arrow[u, two heads, "\pi"'] \end{tikzcd} \quad (2.7)$$

is commutative.

Proof. <u>Existence :</u> We define Φ by induction on the number of vertices. We put $\Phi(1) = 1 \otimes 1$ and $\Phi(\bullet_a) = \varphi(\bullet_a)$ for all $a \in \mathcal{D}$. Suppose that Φ is defined for all forest F of vertices degree $< n$ and satisfies the condition $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F)$. Let $F \in \mathbf{H}_{CK}^{\mathcal{D}}$ be a forest of vertices degree n. If $F = F_1 F_2$, we put $\Phi(F) = \Phi(F_1)\Phi(F_2)$. Suppose that F is a tree. By induction hypothesis, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F)$ is well defined. Moreover,

$$\begin{aligned} & (\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}) \circ (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & (\Phi \otimes \Phi \otimes \Phi) \circ (\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes Id_{\mathbf{H}_{CK}^{\mathcal{D}}} - Id_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & 0, \end{aligned}$$

using induction hypothesis in the first equality and the coassociativity in the second equality.

So $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(F) \in Ker(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$. As $\mathbf{Sh}^{\mathcal{D}}$ is cofree, with lemma 24, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(F) \in Im(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$ and there exists $w \in \mathbf{Sh}^{\mathcal{D}}$ such that $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w)$. We put $\Phi(F) = w - \pi(w) + \varphi(F)$. Then

$$\begin{array}{rcl} \pi \circ \Phi(F) & = & \pi(w) - \pi \circ \pi(w) + \pi \circ \varphi(F) = \varphi(F), \\ \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) & = & \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) - \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\pi(w)) + \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\varphi(F)) \\ & = & \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) \\ & = & (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(F). \end{array}$$

By induction, the result is established.

Uniqueness : Let Φ_1 and Φ_2 be two Hopf algebra morphisms such that the diagram (2.7) is commutative. Let us prove that $\Phi_1(T) = \Phi_2(T)$ for all tree $T \in \mathbf{H}^{\mathcal{D}}_{CK}$ by induction on the vertices degree of T. If $n = 0$, $\Phi_1(1) = \Phi_2(1) = 1$. If $n = 1$, for $i = 1, 2$, $\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_i(\bullet_a) = (\Phi_i \otimes \Phi_i) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(\bullet_a) = 0$. So $\Phi_i(\bullet_a) \in Vect(\mathcal{D})$. As the diagram (2.7) is commutative, $\Phi_1(\bullet_a) = \Phi_2(\bullet_a) = \varphi(\bullet_a)$. Suppose that the result is true in vertices degree $< n$ and let T be a tree of vertices degree n. Using induction hypothesis in the second equality,

$$\begin{array}{rcl} \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_1(T) & = & (\Phi_1 \otimes \Phi_1) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T) \\ & = & (\Phi_2 \otimes \Phi_2) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T) \\ & = & \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_2(T). \end{array}$$

So $\Phi_1(T) - \Phi_2(T) \in Vect(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}})$ and $\Phi_1(T) - \Phi_2(T) = \pi(\Phi_1(T) - \Phi_2(T)) = \varphi(T) - \varphi(T) = 0$. $\square$

Notation. We consider $F \in \mathbf{H}_{CK}$, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(F))$ a linear order on $Cont_{\boldsymbol{e}}(F)$ (see definition 5). For all $i \in \{1, \ldots, |Cont_{\boldsymbol{e}}(F)|_v\}$, $\sigma^{-1}(i)$ is the connected component of $Part_{\boldsymbol{e}}(F)$ such that her image by σ is equal to i.

The following proposition give a combinatorial description of the morphism Φ defined in theorem 25 :

Proposition 26 *Let T be a nonempty tree $\in \mathbf{H}^{\mathcal{D}}_{CK}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(F)|_v)) \ldots \varphi(\sigma^{-1}(1)) \right). \tag{2.8}$$

Proof. We use the following lemma :

Lemma 27 *Let T be a rooted tree of vertices degree n. We define :*

$$\begin{array}{rcl} \mathbb{E}(T) & = & \{(\boldsymbol{v}, \sigma_1, \sigma_2) \mid \boldsymbol{v} \models\!\!\!\models V(T), \sigma_1 \in \mathcal{O}(Lea_{\boldsymbol{v}}(T)), \sigma_2 \in \mathcal{O}(Roo_{\boldsymbol{v}}(T))\}, \\ \mathbb{F}(T) & = & \{(\sigma, p) \mid \sigma \in \mathcal{O}(T), p \in \{1, \ldots, n-1\}\}. \end{array}$$

Then $\mathbb{E}(T)$ and $\mathbb{F}(T)$ are in bijection.

Proof. We define two maps f and g.
Let f be the map defined by

$$f : \left\{ \begin{array}{rcl} \mathbb{E}(T) & \rightarrow & \mathbb{F}(T) \\ (\boldsymbol{v}, \sigma_1, \sigma_2) & \mapsto & (\sigma, |Roo_{\boldsymbol{v}}(T)|_v) \end{array} \right.$$

where $\sigma : V(T) \rightarrow \{1, \ldots, n\}$ is defined by $\sigma(v) = \sigma_2(v)$ for all $v \in V(Roo_{\boldsymbol{v}}(T))$ and $\sigma(v) = \sigma_1(v) + |Roo_{\boldsymbol{v}}(T)|_v$ for all $v \in V(Lea_{\boldsymbol{v}}(T))$. By definition, $\sigma \in \mathcal{O}(T)$.
Let g be the map defined by

$$g : \left\{ \begin{array}{rcl} \mathbb{F}(T) & \rightarrow & \mathbb{E}(T) \\ (\sigma, p) & \mapsto & (\boldsymbol{v}, \sigma_1, \sigma_2) \end{array} \right.$$

where

- $\sigma_1 : V(Lea_{\boldsymbol{v}}(T)) \to \{1, \ldots, |Lea_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_1(v) = \sigma(v) - |Roo_{\boldsymbol{v}}(T)|_v$ for all $v \in V(Lea_{\boldsymbol{v}}(T))$. Then $\sigma_1 \in \mathcal{O}(Lea_{\boldsymbol{v}}(T))$
- $\sigma_2 : V(Roo_{\boldsymbol{v}}(T)) \to \{1, \ldots, |Roo_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_2(v) = \sigma(v)$ for all $v \in V(Roo_{\boldsymbol{v}}(T))$. Then $\sigma_2 \in \mathcal{O}(Roo_{\boldsymbol{v}}(T))$
- $\boldsymbol{v}$ is the subset $\left\{v \in \sigma^{-1}(\{k, \ldots, n\}) \mid \text{if } w \in \sigma^{-1}(\{k, \ldots, n\}) \text{ and } v \twoheadrightarrow w \text{ then } v = w\right\}$ of $V(T)$. We have $\boldsymbol{v} \models V(T)$.

So f and g are well defined. Then we show easily that $f \circ g = Id_{\mathbb{F}(T)}$ and $g \circ f = Id_{\mathbb{E}(T)}$. □

Let us show formula (2.8) by induction on the number n of vertices. If $n = 1$, $T = \bullet_a$ with $a \in \mathcal{D}$. Then $\Phi(\bullet_a) = \varphi(\bullet_a)$ and formula (2.8) is true. If $n \geq 2$,

$$
\begin{aligned}
& \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(T) \\
= \ & (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T) \\
= \ & \sum_{\boldsymbol{v} \models V(T)} \Phi(Lea_{\boldsymbol{v}}(T)) \otimes \Phi(Roo_{\boldsymbol{v}}(T)) \\
= \ & \sum_{\boldsymbol{v} \models V(T)} \left(\sum_{\boldsymbol{e} \models E(Lea_{\boldsymbol{v}}(T))} \left(\sum_{\sigma_1 \in \mathcal{O}(Cont_{\boldsymbol{e}}(Lea_{\boldsymbol{v}}(T)))} \varphi(\sigma_1^{-1}(|Cont_{\boldsymbol{e}}(Lea_{\boldsymbol{v}}(T))|_v)) \ldots \varphi(\sigma_1^{-1}(1)) \right) \right) \\
& \otimes \left(\sum_{\boldsymbol{f} \models E(Roo_{\boldsymbol{v}}(T))} \left(\sum_{\sigma_2 \in \mathcal{O}(Cont_{\boldsymbol{f}}(Roo_{\boldsymbol{v}}(T)))} \varphi(\sigma_2^{-1}(|Cont_{\boldsymbol{f}}(Roo_{\boldsymbol{v}}(T))|_v)) \ldots \varphi(\sigma_2^{-1}(1)) \right) \right) \\
= \ & \sum_{\boldsymbol{e} \models E(T)} \sum_{(\boldsymbol{v}, \sigma_1, \sigma_2) \in \mathbb{E}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma_1^{-1}(|Lea_{\boldsymbol{v}}(Cont_{\boldsymbol{e}}(T))|_v)) \ldots \varphi(\sigma_1^{-1}(1)) \\
& \qquad \otimes \varphi(\sigma_2^{-1}(|Roo_{\boldsymbol{v}}(Cont_{\boldsymbol{e}}(T))|_v)) \ldots \varphi(\sigma_2^{-1}(1)) \\
= \ & \sum_{\boldsymbol{e} \models E(T)} \sum_{(\sigma, p) \in \mathbb{F}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(T)|_v)) \ldots \varphi(\sigma^{-1}(p+1)) \otimes \varphi(\sigma^{-1}(p)) \ldots \varphi(\sigma^{-1}(1)).
\end{aligned}
$$

So

$$
\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(F)|_v)) \ldots \varphi(\sigma^{-1}(1)) \right)
$$

and by induction, we have the result. □

Examples.

- In vertices degree 1, $\Phi(\bullet_a) = \varphi(\bullet_a)$.
- In vertices degree 2,

$$
\begin{aligned}
\Phi(\mathbf{1}^{b}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\mathbf{1}^{b}_{a}) \\
\Phi(\bullet_a \bullet_b) &= \varphi(\bullet_a)\varphi(\bullet_b) + \varphi(\bullet_b)\varphi(\bullet_a).
\end{aligned}
$$

- In vertices degree 3,

$$
\begin{aligned}
\Phi({}^{b}\mathbf{V}^{c}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_b)\varphi(\mathbf{1}^{c}_{a}) + \varphi(\bullet_c)\varphi(\mathbf{1}^{b}_{a}) + \varphi({}^{b}\mathbf{V}^{c}_{a}) \\
\Phi(\mathbf{I}^{c}_{b\,a}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\mathbf{1}^{b}_{a}) + \varphi(\mathbf{1}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{I}^{c}_{b\,a}) \\
\Phi({}^{c}_{b}\mathbf{V}^{d}_{a}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) \\
&\quad + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\mathbf{1}^{d}_{a}) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\mathbf{1}^{b}_{a}) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\mathbf{1}^{b}_{a}) + \varphi(\mathbf{1}^{c}_{b})\varphi(\bullet_d)\varphi(\bullet_a) \\
&\quad + \varphi(\bullet_d)\varphi(\mathbf{1}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{1}^{c}_{b})\varphi(\mathbf{1}^{d}_{a}) + \varphi(\bullet_c)\varphi({}^{b}\mathbf{V}^{d}_{a}) + \varphi(\bullet_d)\varphi(\mathbf{I}^{c}_{b\,a}) + \varphi({}^{c}_{b}\mathbf{V}^{d}_{a}).
\end{aligned}
$$

Particular case. If $\varphi(\bullet_a) = a$ for all $a \in \mathcal{D}$ and $\varphi(T) = 0$ if $|T|_v \geq 1$, then this is the particular case of arborification (see [EV04]). For example :

$\Phi(\bullet_a) = a$	$\Phi(\mathbf{1}^{b}_{a}) = ba$	$\Phi(\bullet_a \bullet_b) = ab + ba$
$\Phi({}^{b}\mathbf{V}^{c}_{a}) = bca + cba$	$\Phi(\mathbf{I}^{c}_{b\,a}) = cba$	$\Phi({}^{c}_{b}\mathbf{V}^{d}_{a}) = cbda + cdba + dcba.$

2.3.2 From $\mathbf{H}^{\mathcal{D}}_{CK}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 28 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}^{\mathcal{D}}_{CK} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd}
\mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\
\mathbf{H}^{\mathcal{D}}_{CK} \arrow[r, "\Phi"] & \mathbf{Csh}^{\mathcal{D}} \arrow[u, two heads, "\pi"']
\end{tikzcd} \tag{2.9}$$

is commutative.

Proof. Noting that $\mathbf{Csh}^{\mathcal{D}}$ is cofree, this is the same proof as for theorem 25. $\square$

Notation. Let $F \in \mathbf{H}_{CK}$ be a nonempty rooted forest, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}_p(Cont_{\boldsymbol{e}}(F))$ a linear preorder on $Cont_{\boldsymbol{e}}(F)$ (see definition 9), $\sigma : V(Cont_{\boldsymbol{e}}(F)) \to \{1,\dots,q\}$ surjective. For all $i \in \{1,\dots,q\}$, $\sigma^{-1}(i)$ is the forest $T_1 \dots T_n$ of all connected components T_k of $Part_{\boldsymbol{e}}(F)$ such that $\sigma(T_k) = i$ for all $k \in \{1,\dots,n\}$. In this case, $\varphi(\sigma^{-1}(i))$ is the element $[\varphi(T_1)\dots\varphi(T_n)]^{(n)}$.

Now, we give a combinatorial description of the morphism Φ defined in theorem 28 :

Proposition 29 *Let T be a nonempty tree $\in \mathbf{H}^{\mathcal{D}}_{CK}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\substack{\sigma \in \mathcal{O}_p(Cont_{\boldsymbol{e}}(T)) \\ Im(\sigma)=\{1,\dots,q\}}} \varphi(\sigma^{-1}(q)) \dots \varphi(\sigma^{-1}(1)) \right). \tag{2.10}$$

Proof. It suffices to resume the proof of proposition 26. Note that, if T is a rooted tree and $\boldsymbol{v} \models V(T)$, $Roo_{\boldsymbol{v}}(T)$ is a tree and $Lea_{\boldsymbol{v}}(T)$ is a forest. So there is possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$. We deduce formula (2.10). $\square$

Examples.
- In vertices degree 1, $\Phi(\bullet_a) = \varphi(\bullet_a)$.
- In vertices degree 2,

$$\begin{aligned}
\Phi(\bullet_a \bullet_b) &= \varphi(\bullet_a)\varphi(\bullet_b) + \varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_a)\varphi(\bullet_b)] \\
\Phi(\mathbf{I}^{b}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\mathbf{I}^{b}_{a}).
\end{aligned}$$

- In vertices degree 3,

$$\begin{aligned}
\Phi({}^{b}\mathbf{V}^{c}_{a}) &= \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_c)\varphi(\bullet_b)]\,\varphi(\bullet_a) + \varphi(\bullet_b)\varphi(\mathbf{I}^{c}_{a}) \\
&\quad + \varphi(\bullet_c)\varphi(\mathbf{I}^{b}_{a}) + \varphi({}^{b}\mathbf{V}^{c}_{a}) \\
\Phi(\mathbf{I}^{c}_{\substack{b\\a}}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\mathbf{I}^{b}_{a}) + \varphi(\mathbf{I}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{I}^{c}_{\substack{b\\a}}) \\
\Phi({}^{c}_{b}\mathbf{V}^{d}_{a}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_c)\,[\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a) \\
&\quad + [\varphi(\bullet_c)\varphi(\bullet_d)]\,\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\mathbf{I}^{d}_{a}) \\
&\quad + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\mathbf{I}^{b}_{a}) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\mathbf{I}^{b}_{a}) + \varphi(\mathbf{I}^{c}_{b})\varphi(\bullet_d)\varphi(\bullet_a) \\
&\quad + [\varphi(\mathbf{I}^{c}_{b})\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\mathbf{I}^{c}_{b})\varphi(\bullet_a) + \varphi(\mathbf{I}^{c}_{b})\varphi(\mathbf{I}^{d}_{a}) + \varphi(\bullet_c)\varphi({}^{b}\mathbf{V}^{d}_{a}) \\
&\quad + \varphi(\bullet_d)\varphi(\mathbf{I}^{c}_{\substack{b\\a}}) + \varphi({}^{c}_{b}\mathbf{V}^{d}_{a}).
\end{aligned}$$

Particular case. If $\varphi(\bullet_a) = a$ for all $a \in \mathcal{D}$ and $\varphi(T) = 0$ if $|T|_v \geq 1$, then this is the particular case of contracting arborification (see [EV04]). For example :

$\Phi(\bullet_a) = a$	$\Phi(\mathbf{I}^{b}_{a}) = ba$	$\Phi(\bullet_a \bullet_b) = ab + ba + [ab]$
$\Phi({}^{b}\mathbf{V}^{c}_{a}) = bca + cba + [bc]a$	$\Phi(\mathbf{I}^{c}_{\substack{b\\a}}) = cba$	$\Phi({}^{c}_{b}\mathbf{V}^{d}_{a}) = cbda + cdba + dcba + c[bd]a + [cd]ba.$

2.3.3 From $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}\left(\mathcal{D}\right)$ be a $\mathbb{K}$-linear map.

Theorem 30 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}_{CK}^{\mathcal{D}} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}\left(\mathcal{D}\right) \\ \mathbf{C}_{CK}^{\mathcal{D}} \arrow[r, "\Phi"] & \mathbf{Sh}^{\mathcal{D}} \arrow[u, two heads, "\pi"'] \end{tikzcd} \tag{2.11}$$

is commutative.

Proof. This is the same proof as for theorem 25. □

As in the sections 2.3.1 and 2.3.2, we give a combinatorial description of the morphism Φ defined in theorem 30. We need the following definition :

Definition 31 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized partition of F is a k-uplet $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k)$ of subsets of $E(F)$, $1 \leq k \leq |F|_e$, such that :*

1. *$\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ if $i \neq j$ and $\cup_i \boldsymbol{e}_i = E(F)$,*
2. *the edges of any $\boldsymbol{e}_i$ belong to the same connected component of F,*
3. *if v and w are two vertices of $Part_{\boldsymbol{e}_i}(F)$ and if the shortest path in F between v and w contains an edge $\in \boldsymbol{e}_j$, then $j < i$.*

We shall denote by $\mathcal{P}(F)$ the set of generalized partitions of F.

Proposition 32 *Let F be a nonempty forest $\in \mathbf{C}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(F) = \sum_{(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\in\mathcal{P}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)). \tag{2.12}$$

Proof. We use the following lemma :

Lemma 33 *If $F \in \mathbf{C}_{CK}^{\mathcal{D}}$ is a nonempty tree, then the sets*

$$\begin{array}{rcl} \mathbb{E}(F) & = & \{((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p) \mid (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F), 1 \leq p \leq k-1\} \\ \mathbb{F}(F) & = & \{\left(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\right) \mid \boldsymbol{e} \Vdash E(F), (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F)), \\ & & (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))\} \end{array}$$

are in bijection.

Proof. Consider the following two maps :

$$f : \left\{ \begin{array}{rcl} \mathbb{E}(F) & \to & \mathbb{F}(F) \\ ((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p) & \mapsto & (\cup_{1\leq i\leq p}\boldsymbol{e}_i, (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p), (\boldsymbol{e}_{p+1}, \ldots, \boldsymbol{e}_k)) \end{array} \right.$$

and

$$g : \left\{ \begin{array}{rcl} \mathbb{F}(F) & \to & \mathbb{E}(F) \\ \left(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\right) & \mapsto & \left((\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r), q\right). \end{array} \right.$$

<u>f is well defined :</u>

Let $((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p) \in \mathbb{E}(F)$. Then $\boldsymbol{e} = \cup_{1\leq i\leq p}\boldsymbol{e}_i$ is a nonempty nontotal contraction of F.

1. (a) $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p)$ is a p-uplet of subsets of $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$. By hypothesis, $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$. So $\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ and $\cup_{1\leq i\leq p}\boldsymbol{e}_i = E(Part_{\boldsymbol{e}}(F))$.
   (b) The edges $\in \boldsymbol{e}_i$, $1 \leq i \leq p$, are the edges of the same connected component of F therefore of $Part_{\boldsymbol{e}}(F)$ because $\boldsymbol{e}_i \subseteq \boldsymbol{e}$.

(c) Let v and w be two vertices of $Part_{\boldsymbol{e}_i}(Part_{\boldsymbol{e}}(F)) = Part_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i \subseteq \boldsymbol{e}$). If the shortest path in $Part_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in \boldsymbol{e}_j$, then the shortest path in F between v and w contains also an edge $\in \boldsymbol{e}_j$. As $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$, we have $j < i$.

So $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$.

2. (a) $(\boldsymbol{e}_{p+1}, \ldots, \boldsymbol{e}_k)$ is a $(k-p)$-uplet of subsets of $E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$. By hypothesis, $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$. So $\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ and $\cup_{p+1 \leq i \leq k} \boldsymbol{e}_i = E(Cont_{\boldsymbol{e}}(F))$.

   (b) The edges $\in \boldsymbol{e}_i$, $p+1 \leq i \leq k$, are the edges of the same connected component of F therefore of $Cont_{\boldsymbol{e}}(F)$ (we contract in F some connected components).

   (c) Let i be an integer $\in \{p+1, \ldots, k\}$ and v and w two vertices of $Part_{\boldsymbol{e}_i}(Cont_{\boldsymbol{e}}(F)) = Part_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i \cap \boldsymbol{e} = \emptyset$). If the shortest path in $Cont_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in \boldsymbol{e}_j$ then the shortest path in F between v and w contains also an edge $\in \boldsymbol{e}_j$. As $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$, we have $j < i$.

   Thus $(\boldsymbol{e}_{p+1}, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$.

So $f\left(((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p)\right) \in \mathbb{F}(F)$.

<u>g is well defined :</u>

Let $\left(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\right) \in \mathbb{F}(F)$. Let us show that $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(F)$.

1. As $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$ and $(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$, $\boldsymbol{f}_i \neq \emptyset$, $\boldsymbol{g}_i \neq \emptyset$, $\boldsymbol{f}_i \cap \boldsymbol{f}_j = \emptyset$, $\boldsymbol{g}_i \cap \boldsymbol{g}_j = \emptyset$ and $(\cup_i \boldsymbol{f}_i) \bigcup (\cup_i \boldsymbol{g}_i) = E(Part_{\boldsymbol{e}}(F)) \bigcup E(Cont_{\boldsymbol{e}}(F)) = E(F)$. In addition, as $\boldsymbol{f}_i \subseteq E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $\boldsymbol{g}_j \subseteq E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$, $\boldsymbol{f}_i \cap \boldsymbol{g}_j = \emptyset$.

2. The edges $\in \boldsymbol{f}_i$ are the edges of the same connected component of $Part_{\boldsymbol{e}}(F)$. As all the trees of the forest $Part_{\boldsymbol{e}}(F)$ are subtrees of F, the edges $\in \boldsymbol{f}_i$ are the edges of the same connected component of F. Moreover if the edges $\in \boldsymbol{g}_i$ are the edges of the same connected component of $Cont_{\boldsymbol{e}}(F)$, it is also true in F.

3. (a) Let i be an integer $\in \{1, \ldots, q\}$ and v and w two vertices of $Part_{\boldsymbol{f}_i}(F)$. We have $\boldsymbol{f}_i \subseteq \boldsymbol{e}$ therefore $Part_{\boldsymbol{f}_i}(F) = Part_{\boldsymbol{f}_i}(Part_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains :

      i. an edge $\in \boldsymbol{f}_j$. As $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$, $j < i$.

      ii. an edge $\in \boldsymbol{g}_j$. Then the connected component of $Part_{\boldsymbol{e}}(F)$ containing v and w has an edge $\in \boldsymbol{g}_j$. This is impossible because $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $\boldsymbol{g}_j \subseteq E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$.

   (b) Let i be an integer $\in \{1, \ldots, r\}$ and v and w two vertices of $Part_{\boldsymbol{g}_i}(F)$. $\boldsymbol{g}_i \cap \boldsymbol{e} = \emptyset$ therefore $Part_{\boldsymbol{g}_i}(F) = Part_{\boldsymbol{g}_i}(Cont_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains :

      i. an edge $\in \boldsymbol{g}_j$. As $(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$, $j < i$.

      ii. an edge $\in \boldsymbol{f}_j$. It is good because $\boldsymbol{f}_j$ is before $\boldsymbol{g}_i$.

   Thus $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(F)$.

So $g\left(\left(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\right)\right) \in \mathbb{E}(F)$

Finally, we easily see that $f \circ g = Id_{\mathbb{F}(F)}$ and $g \circ f = Id_{\mathbb{E}(F)}$. $\square$

We now prove proposition 32. By induction on the edges degree n of $F \in \mathbf{C}^{\mathcal{D}}_{CK}$. If $n = 1$, $F = \,\substack{\bullet\\|\\\bullet}{}^{a}$ with $a \in \mathcal{D}$. Then $\Phi(\substack{\bullet\\|\\\bullet}{}^{a}) = \varphi(\substack{\bullet\\|\\\bullet}{}^{a})$ and formula (2.12) is true. Suppose that $n \geq 2$ and that the property is true in degrees $k < n$. Then

$$
\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= (\Phi \circ \Phi) \circ \tilde{\Delta}_{\mathbf{C}^{\mathcal{D}}_{CK}}(F) \\
&= \sum_{\boldsymbol{e} \| \models E(F)} \Phi\left(Part_{\boldsymbol{e}}(F)\right) \otimes \Phi\left(Cont_{\boldsymbol{e}}(F)\right) \\
&= \sum_{\boldsymbol{e} \| \models E(F)} \left(\sum_{(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))} \varphi(Cont_{\overline{\boldsymbol{f}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{f}_q}}(F)) \right) \\
&\qquad\qquad \otimes \left(\sum_{(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))} \varphi(Cont_{\overline{\boldsymbol{g}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{g}_r}}(F)) \right)
\end{aligned}
$$

using induction hypothesis in the last equality. So, with lemma 33,

$$\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= \sum_{(\boldsymbol{e},(\boldsymbol{f}_1,\ldots,\boldsymbol{f}_q),(\boldsymbol{g}_1,\ldots,\boldsymbol{g}_r))\in\mathbb{F}(F)} \varphi(Cont_{\overline{\boldsymbol{f}_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{f}_q}}(F)) \\
&\qquad\qquad \otimes\varphi(Cont_{\overline{\boldsymbol{g}_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{g}_r}}(F)) \\
&= \sum_{((\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k),p)\in\mathbb{E}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad\qquad \otimes\varphi(Cont_{\overline{\boldsymbol{e}_{p+1}}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)) \\
&= \sum_{(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\in\mathcal{P}(F)} \sum_{1\leq p\leq k-1} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad\qquad \otimes\varphi(Cont_{\overline{\boldsymbol{e}_{p+1}}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)).
\end{aligned}$$

Therefore

$$\Phi(F) = \sum_{(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\in\mathcal{P}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}_k}}(F))$$

and by induction, we have the result. □

Examples. We introduce a notation. If $w = w_1 \ldots w_n$ is a $\mathcal{D}$-word, we denote $\mathrm{Perm}(w)$ the sum of all $\mathcal{D}$-words whose letters are $w_1, \ldots, w_n$. For example, $\mathrm{Perm}(abc) = abc + acb + bac + bca + cab + cba$.

- In edges degree 1, $\Phi(\mathbf{l}_a) = \varphi(\mathbf{l}_a)$.
- In edges degree 2,

$$\begin{aligned}
\Phi(\mathbf{V}_{a,b}) &= \varphi(\mathbf{V}_{a,b}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b) + \varphi(\mathbf{l}_b)\varphi(\mathbf{l}_a) \\
\Phi(\mathbf{l}_a^b) &= \varphi(\mathbf{l}_a^b) + \varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b) + \varphi(\mathbf{l}_b)\varphi(\mathbf{l}_a).
\end{aligned}$$

- In edges degree 3,

$$\begin{aligned}
\Phi(\mathbf{W}_{a,b,c}) &= \varphi(\mathbf{W}_{a,b,c}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,c}) + \varphi(\mathbf{V}_{b,c})\varphi(\mathbf{l}_a) + \varphi(\mathbf{l}_b)\varphi(\mathbf{V}_{a,c}) + \varphi(\mathbf{V}_{a,c})\varphi(\mathbf{l}_b) \\
&\quad + \varphi(\mathbf{l}_c)\varphi(\mathbf{V}_{a,b}) + \varphi(\mathbf{V}_{a,b})\varphi(\mathbf{l}_c) + \mathrm{Perm}(\varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_c)) \\
\Phi(\mathbf{Y}_a^{b,c}) &= \varphi(\mathbf{Y}_a^{b,c}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,c}) + \varphi(\mathbf{V}_{b,c})\varphi(\mathbf{l}_a) + \varphi(\mathbf{l}_c)\varphi(\mathbf{l}_a^b) + \varphi(\mathbf{l}_a^b)\varphi(\mathbf{l}_c) \\
&\quad + \varphi(\mathbf{l}_b)\varphi(\mathbf{l}_a^c) + \varphi(\mathbf{l}_a^c)\varphi(\mathbf{l}_b) + \mathrm{Perm}(\varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_c)) \\
\Phi(\mathbf{V}_{a,b}^{c}) &= \varphi(\mathbf{V}_{a,b}^{c}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{c,b}) + \varphi(\mathbf{l}_b)\varphi(\mathbf{l}_a^c) + \varphi(\mathbf{l}_a^c)\varphi(\mathbf{l}_b) + \varphi(\mathbf{l}_c)\varphi(\mathbf{V}_{a,b}) \\
&\quad + \varphi(\mathbf{V}_{a,b})\varphi(\mathbf{l}_c) + \mathrm{Perm}(\varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_c)).
\end{aligned}$$

- Finally, in edges degree 4, with the tree $\mathbf{V}_{a,b}^{c,d}$,

$$\begin{aligned}
\Phi(\mathbf{V}_{a,b}^{c,d}) &= \varphi(\mathbf{V}_{a,b}^{c,d}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{W}_{c,d,b}) + \varphi(\mathbf{l}_b)\varphi(\mathbf{Y}_a^{c,d}) + \varphi(\mathbf{Y}_a^{c,d})\varphi(\mathbf{l}_b) + \varphi(\mathbf{l}_c)\varphi(\mathbf{V}_{a,b}^{d}) \\
&\quad + \varphi(\mathbf{V}_{a,b}^{d})\varphi(\mathbf{l}_c) + \varphi(\mathbf{l}_d)\varphi(\mathbf{V}_{a,b}^{c}) + \varphi(\mathbf{V}_{a,b}^{c})\varphi(\mathbf{l}_d) + \varphi(\mathbf{V}_{a,b})\varphi(\mathbf{V}_{c,d}) \\
&\quad + \varphi(\mathbf{V}_{c,d})\varphi(\mathbf{V}_{a,b}) + \varphi(\mathbf{l}_a^d)\varphi(\mathbf{V}_{c,b}) + \varphi(\mathbf{l}_a^c)\varphi(\mathbf{V}_{d,b}) + \mathrm{Perm}(\varphi(\mathbf{V}_{a,b})\varphi(\mathbf{l}_c)\varphi(\mathbf{l}_d)) \\
&\quad + \mathrm{Perm}(\varphi(\mathbf{V}_{c,d})\varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b)) + \mathrm{Perm}(\varphi(\mathbf{l}_a^d)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_c)) \\
&\quad + \mathrm{Perm}(\varphi(\mathbf{l}_a^c)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_d)) + \varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,c})\varphi(\mathbf{l}_d) + \varphi(\mathbf{l}_a)\varphi(\mathbf{l}_d)\varphi(\mathbf{V}_{b,c}) \\
&\quad + \varphi(\mathbf{l}_d)\varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,c}) + \varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,d})\varphi(\mathbf{l}_c) + \varphi(\mathbf{l}_a)\varphi(\mathbf{l}_c)\varphi(\mathbf{V}_{b,d}) \\
&\quad + \varphi(\mathbf{l}_c)\varphi(\mathbf{l}_a)\varphi(\mathbf{V}_{b,d}) + \mathrm{Perm}(\varphi(\mathbf{l}_a)\varphi(\mathbf{l}_b)\varphi(\mathbf{l}_c)\varphi(\mathbf{l}_d))
\end{aligned}$$

2.3.4 From $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 34 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}^{\mathcal{D}}_{CK} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{tikzcd} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \arrow[r, "\varphi"] \arrow[d, hook, "i"] & \mathbb{K}(\mathcal{D}) \\ \mathbf{C}^{\mathcal{D}}_{CK} \arrow[r, "\Phi"] & \mathbf{Csh}^{\mathcal{D}} \arrow[u, two heads, "\pi"] \end{tikzcd} \tag{2.13}$$

is commutative.

Proof. This is the same proof as for theorem 25. □

We give a combinatorial description of the morphism Φ defined in theorem 34. For this, we give the following definition :

Definition 35 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized and contracted partition of F is a l-uplet $(f_1, \ldots, f_l)$ such that :*

1. *for all $1 \leq i \leq l$, $f_i = (\boldsymbol{e}^i_1, \ldots, \boldsymbol{e}^i_{k_i})$ is a k_i-uplet of subsets of $E(F)$,*
2. *$(\boldsymbol{e}^1_1, \ldots, \boldsymbol{e}^1_{k_1}, \boldsymbol{e}^2_1, \ldots, \boldsymbol{e}^l_{k_l}) \in \mathcal{P}(F)$,*
3. *if $Part_{\boldsymbol{e}^i_p}(F)$ and $Part_{\boldsymbol{e}^i_q}(F)$ are two disconnected components of F and if the shortest path in F between $Part_{\boldsymbol{e}^i_p}(F)$ and $Part_{\boldsymbol{e}^i_q}(F)$ contains an edge $\in \boldsymbol{e}^j_r$, then $j > i$.*

We shall denote by $\mathcal{P}_c(F)$ the set of generalized and contracted partitions of F.

Proposition 36 *Let F be a nonempty forest $\in \mathbf{C}^{\mathcal{D}}_{CK}$. Then*

$$\begin{aligned} \Phi(F) \quad = \quad \sum_{\substack{(f_1,\ldots,f_l)\in\mathcal{P}_c(F) \\ f_i=(\boldsymbol{e}^i_1,\ldots,\boldsymbol{e}^i_{k_i})}} & \left(\left[\varphi(Cont_{\overline{\boldsymbol{e}^1_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}^1_{k_1}}}(F))\right]^{(k_1)}\ldots\right. \\ & \left.\ldots\left[\varphi(Cont_{\overline{\boldsymbol{e}^l_1}}(F))\ldots\varphi(Cont_{\overline{\boldsymbol{e}^l_{k_l}}}(F))\right]^{(k_l)}\right). \end{aligned} \tag{2.14}$$

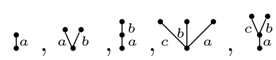
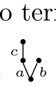
Proof. It suffices to resume the proof of proposition 32. Note that, if T is a rooted tree and $\boldsymbol{e} \models E(T)$, $Cont_{\boldsymbol{e}}(T)$ is a tree and $Part_{\boldsymbol{e}}(T)$ is a forest. So there is possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$. Remark that

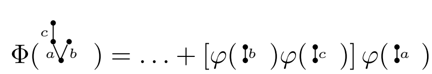
- the trees of $Part_{\boldsymbol{e}}(T)$ are disconnected components of T and they appear to the left of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$,
- the edges of $\overline{\boldsymbol{e}}$ between two disconnected components of $Part_{\boldsymbol{e}}(T)$ in T are edges of $Cont_{\boldsymbol{e}}(T)$ and thus they appear to the right of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$.

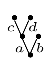
We deduce formula (2.14). □

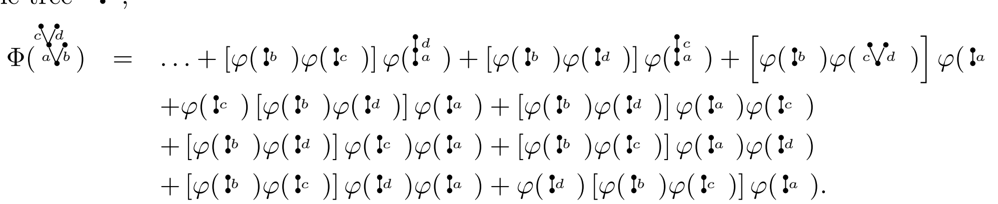
Remark. In the expression of $\Phi(F)$ (formula (2.14)), we find the terms of (2.12) and other terms with contractions for the product $[\cdot,\cdot]$. Taking $[\cdot,\cdot] = 0$, we obtain (2.12) again.

Examples. From the examples at the end of section 2.3.3, we give the other terms with contractions for the product $[\cdot,\cdot]$.

- There are no terms with contractions for the following trees :  .
- For the tree  ,



- For the tree ,



Chapitre 3

On the operad of bigraft algebras

Introduction

Dans ce chapitre, nous introduisons la notion d'algèbres bigreffes. Une algèbre bigreffe est un $\mathbb{K}$-espace vectoriel A muni de trois opérations $*, \succ, \prec \colon A \otimes A \to A$ vérifiant les relations suivantes :

$$\begin{align} (x * y) \succ z &= x \succ (y \succ z), \tag{3.1}\\ (x \succ y) * z &= x \succ (y * z), \tag{3.2}\\ (x \prec y) \prec z &= x \prec (y * z), \tag{3.3}\\ (x * y) \prec z &= x * (y \prec z), \tag{3.4}\\ (x \succ y) \prec z &= x \succ (y \prec z), \tag{3.5}\\ (x * y) * z &= x * (y * z). \tag{3.6} \end{align}$$

Nous proposons dans ce chapitre une étude algébrique de l'opérade bigreffe, noté $\mathcal{BG}$, associée aux algèbres bigreffes. Nous allons décrire cette opérade, montrer qu'elle est Koszul et établir un nouveau bon triplet d'opérades.

Pour cela, nous rappelons tout d'abord les notions connues sur les algèbres de greffe à gauche et à droite. Une algèbre de greffe à droite est un $\mathbb{K}$-espace vectoriel A muni de deux opérations $*, \prec \colon A \otimes A \to A$ vérifiant les relations (3.3), (3.4) et (3.6). Avec les relations (3.1), (3.2) and (3.6), on obtient la notion d'algèbre de greffe à gauche. L. Foissy prouve dans [Foi10] que l'opérade des algèbres de greffe à droite est donnée en termes de forêts enracinées planes.

Nous prouvons ici que l'opérade bigreffe peut être décrite en termes d'un sous-ensemble des forêts enracinées planes dont les arêtes sont décorées (on décore les arêtes avec un ensemble de décorations réduit à deux éléments). Ce sous-ensemble est défini récursivement à l'aide d'un opérateur de greffe B_{BG}. On notera $\mathbf{H}_{BG}$ l'algèbre engendrée par ce sous-ensemble.

Nous donnons une présentation du dual de Koszul $\mathcal{BG}^!$ de l'opérade $\mathcal{BG}$ (voir [GK94, MSS02]). En considérant un quotient de $\mathbf{H}_{BG}$, nous décrivons l'opérade $\mathcal{BG}^!$ et cette description nous permet de construire l'homologie associée à une $\mathcal{BG}$-algèbre. A l'aide d'une méthode de réécriture (voir [DK10, Hof10, LV12]), on prouve que $\mathcal{BG}$ est Koszul et nous donnons des bases de PBW de l'opérade bigreffe et de son dual de Koszul.

Ensuite, nous prouvons que l'opérateur de greffe B_{BG} introduit précédemment induit une structure d'algèbre de Hopf sur $\mathbf{H}_{BG}$ et permet aussi de munir $\mathbf{H}_{BG}$ d'un couplage de Hopf. Nous décrivons les relations de compatibilités ce coproduit et les produits $*, \succ, \prec$.

Comme ces relations ne permettent pas de définir une bonne notion de bialgèbre bigreffe, nous considérons un autre coproduit sur $\mathbf{H}_{BG}$, le coproduit de déconcaténation $\tilde{\Delta}_{\mathcal{A}ss}$. Nous prouvons que l'idéal d'augmentation de $\mathbf{H}_{BG}$ est une bialgèbre bigreffe infinitésimale : c'est une famille $(A, *, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ où $*, \succ, \prec \colon A \otimes A \to A$, $\tilde{\Delta}_{\mathcal{A}ss} : A \to A \otimes A$, telle que $(A, *, \succ, \prec)$ est une algèbre bigreffe et pour tout $x, y \in A$:

$$\left\{ \begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x * y) &=& (x \otimes 1) * \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x) * (1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) &=& (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) &=& \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array} \right.$$

Nous montrons alors des versions analogues aux théorèmes de Poincaré-Birkhoff-Witt et de Cartier-Milnor-Moore dans le cas des bialgèbres bigreffes infinitésimales (en utilisant les résultats de [Lod08]).

Nous prouvons que la partie primitive $Prim(A)$ d'une bialgèbre bigreffe infinitésimale A est une $\mathcal{L}$-algèbre, c'est-à-dire un $\mathbb{K}$-espace vectoriel muni de deux opérations binaires $\succ, \prec$ vérifiant la relation d'entrelacement (3.5) (voir [Ler03, Ler08] pour plus de détails sur les $\mathcal{L}$-algèbres). Nous définissons l'algèbre bigreffe enveloppante universelle $U_{\mathcal{BG}}(A)$ d'une $\mathcal{L}$-algèbre A et on prouve alors le résultat suivant :

Theorem. *Si A est une bialgèbre bigreffe infinitésimale sur un corps $\mathbb{K}$, alors les assertions suivantes sont équivalentes :*

1. *A est une bialgèbre bigreffe infinitésimale connexe,*
2. *A est colibre parmi les coalgèbres connexes,*
3. *A est isomorphe à $U_{\mathcal{BG}}(Prim(A))$ comme bialgèbre bigreffe infinitésimale.*

On déduit de ce théorème un nouveau bon triplet d'opérades $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ faisant intervenir les opérades associées aux algèbres associatives, bigreffes et aux $\mathcal{L}$-algèbres.

Ce chapitre est organisé comme suit : dans une première partie, nous définissons la notion d'algèbre bigreffe et nous rappelons différents résultats sur les algèbres de greffes à gauche et à droite. Nous donnons une description combinatoire de l'algèbre bigreffe libre et de l'opérade bigreffe. La partie 2 est consacrée à l'étude du dual de Koszul de l'opérade bigreffe. Nous déterminons l'opérade duale et nous montrons que l'opérade bigreffe est Koszul. Dans la troisième partie, nous munissons l'algèbre bigreffe libre à un générateur d'une structure d'algèbre de Hopf et d'un couplage. Nous introduisons dans la dernière partie la notion de bialgèbres bigreffes infinitésimales et nous démontrons que $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ est un bon triplet d'opérades.

Les résultats de ce chapitre sont rassemblés dans un article intitulé *On the operad of bigraft algebras* à paraître dans Journal of Algebraic Combinatorics.

Remarque. Dans ce chapitre, le degré d'un arbre ou d'une forêt est le degré en sommets.

3.1 Operad of bigraft algebras

3.1.1 Bigraft algebras

Recall the definitions of left graft algebras and right graft algebras introduced by L. Foissy (see [Foi10]) :

Definition 37 1. *A left graft algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $*, \succ$: $A \otimes A \to A$ respectively called product and left graft, satisfying the following relations : for all $x, y, z \in A$,*

$$\begin{aligned}(x * y) * z &= x * (y * z),\\ (x * y) \succ z &= x \succ (y \succ z),\\ (x \succ y) * z &= x \succ (y * z).\end{aligned}$$

2. *The left graft operad, denoted by $\mathcal{LG}$, is the operad such that $\mathcal{LG}$-algebras are left graft algebras.*

Definition 38 1. *A right graft algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $*, \prec$: $A \otimes A \to A$ respectively called product and right graft, satisfying the following relations : for all $x, y, z \in A$,*

$$\begin{aligned}(x * y) * z &= x * (y * z),\\ (x \prec y) \prec z &= x \prec (y * z),\\ (x * y) \prec z &= x * (y \prec z).\end{aligned}$$

2. *The right graft operad, denoted by $\mathcal{RG}$, is the operad such that $\mathcal{RG}$-algebras are right graft algebras.*

It is clear that the operads $\mathcal{LG}$ and $\mathcal{RG}$ are binary, quadratic, regular and set-theoretic (see [LV12] for a definition). It is proved that $\mathcal{LG}$ and $\mathcal{RG}$ are Koszul in [Foi09b]. We do not suppose that $\mathcal{LG}$-algebras and $\mathcal{RG}$-algebras have units for the product $*$. If A and B are two $\mathcal{LG}$-algebras, we say that a $\mathbb{K}$-linear map $f : A \to B$ is a $\mathcal{LG}$-morphism if $f(x * y) = f(x) * f(y)$ and $f(x \succ y) = f(x) \succ f(y)$ for all $x, y \in A$.

We define in the same way the notion of $\mathcal{RG}$-morphism. We denote by $\mathcal{LG}$-alg the category of $\mathcal{LG}$-algebras and $\mathcal{RG}$-alg the category of $\mathcal{RG}$-algebras.

Remark. The category $\mathcal{LG}$-alg is equivalent to the category $\mathcal{RG}$-alg : let $(A, *, \succ)$ be a $\mathcal{LG}$-algebra, then $(A, *^{\dagger}, \succ^{\dagger})$ is a $\mathcal{RG}$-algebra, where $x *^{\dagger} y = y * x$ and $x \succ^{\dagger} y = y \succ x$ for all $x, y \in A$. Note that $*^{\dagger\dagger} = *$ and $\succ^{\dagger\dagger} = \succ$. So we will only study the operad $\mathcal{RG}$.

We now give the definition of bigraft algebras :

Definition 39 *1. A bigraft algebra is a $\mathbb{K}$-vector space A together with three $\mathbb{K}$-linear maps $*, \succ, \prec : A \otimes A \to A$ satisfying the following relations : for all $x, y, z \in A$,*

$$\begin{array}{rcl} (x * y) \succ z & = & x \succ (y \succ z), \\ (x \succ y) * z & = & x \succ (y * z), \\ (x \prec y) \prec z & = & x \prec (y * z), \\ (x * y) \prec z & = & x * (y \prec z), \\ (x \succ y) \prec z & = & x \succ (y \prec z), \\ (x * y) * z & = & x * (y * z). \end{array} \tag{3.7}$$

2. The bigraft operad, denoted by $\mathcal{BG}$, is the operad such that $\mathcal{BG}$-algebras are bigraft algebras. $\mathcal{BG}$ is a binary, quadratic, regular and set-theoretic operad. We denote by $\widetilde{\mathcal{BG}}$ the nonsymmetric operad associated with the regular operad $\mathcal{BG}$.

In other words, a $\mathcal{BG}$-algebra $(A, *, \succ, \prec)$ is a $\mathcal{LG}$-algebra $(A, *, \succ)$ and a $\mathcal{RG}$-algebra $(A, *, \prec)$ verifying the so-called entanglement relation $(x \succ y) \prec z = x \succ (y \prec z)$ for all $x, y, z \in A$, that is to say $(A, \succ, \prec)$ is a $\mathcal{L}$-algebra (see definition 79). We do not suppose that $\mathcal{BG}$-algebras have a unit for the product $*$. If A and B are two $\mathcal{BG}$-algebras, a $\mathcal{BG}$-morphism from A to B is a $\mathbb{K}$-linear map $f : A \to B$ such that $f(x * y) = f(x) * f(y)$, $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by $\mathcal{BG}$-alg the category of $\mathcal{BG}$-algebras.

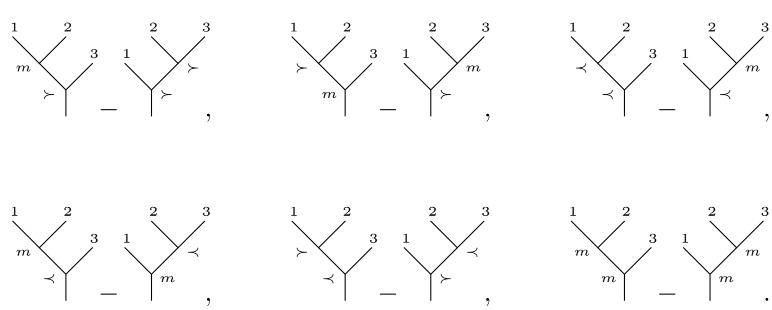
Remark. Let $(A, *, \succ, \prec)$ be a $\mathcal{BG}$-algebra. Then $(A, *^{\dagger}, \prec^{\dagger}, \succ^{\dagger})$ is a $\mathcal{BG}$-algebra, where $x *^{\dagger} y = y * x$, $x \succ^{\dagger} y = y \prec x$ and $x \prec^{\dagger} y = y \succ x$ for all $x, y \in A$.

From definition 39, we give a description of the operad $\mathcal{BG}$ by generators and relations :

Lemma 40 *The operad $\mathcal{BG}$ is the quadratic operad generated by three operations denoted by m, $\succ$ and $\prec$ and satisfying the following relations, where I is the unit element of $\mathcal{BG}$:*

$$\begin{cases} \succ \circ (m, I) - \succ \circ (I, \succ), \\ m \circ (\succ, I) - \succ \circ (I, m), \\ \prec \circ (\prec, I) - \prec \circ (I, m), \\ \prec \circ (m, I) - m \circ (I, \prec), \\ \prec \circ (\succ, I) - \succ \circ (I, \prec), \\ m \circ (m, I) - m \circ (I, m). \end{cases}$$

Remark. Graphically, the relations defining $\mathcal{BG}$ can be written in the following way :



3.1.2 Free pre-Lie algebra and free right graft algebra

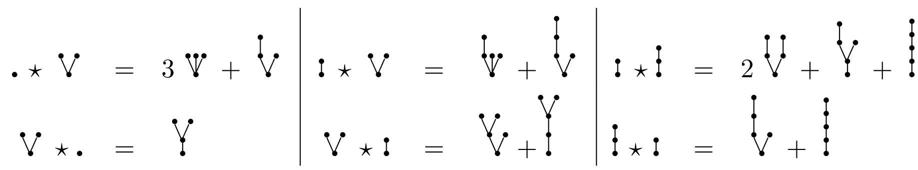
We here recall some results of [CL01, Foi10].

Let $\mathbf{T}_{CK}$ be the $\mathbb{K}$-vector space spanned by $\mathbb{T}_{\mathbf{H}_{CK}}$ the set of the nonempty rooted trees.

If $T, T_1, T_2 \in \mathbb{T}_{\mathbf{H}_{CK}}$, we denote by $n_{(T_1,T_2,T)}$ the number of admissible cuts $\boldsymbol{v}$ such that $Lea_{\boldsymbol{v}}(T) = T_1$ and $Roo_{\boldsymbol{v}}(T) = T_2$. We consider the linear map $\star : \mathbf{T}_{CK} \otimes \mathbf{T}_{CK} \to \mathbf{T}_{CK}$ such that, if $T_1, T_2 \in \mathbb{T}_{\mathbf{H}_{CK}}$, then

$$T_1 \star T_2 = \sum_{T \in \mathbb{T}_{\mathbf{H}_{CK}}} n_{(T_1,T_2,T)} T.$$

Examples.



Let us recall that a pre-Lie algebra is a $\mathbb{K}$-vector space A equipped with a binary operation $\star : A \otimes A \to A$ such that, for all $x, y, z \in A$,

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y).$$

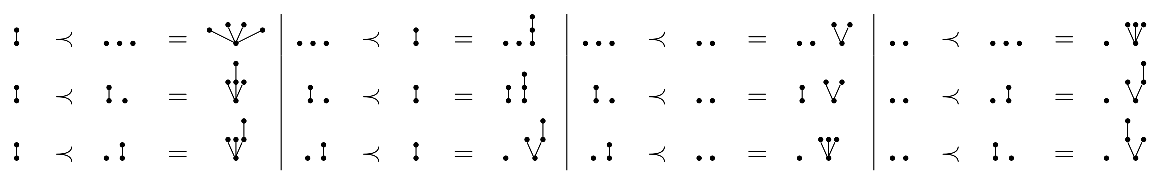
It is proved in [CL01] that $(\mathbf{T}_{CK}, \star)$ is the free pre-Lie algebra generated by $\bullet$.

Let $\mathbf{M}_{NCK}$ be the $\mathbb{K}$-vector space spanned by the nonempty forests of $\mathbf{H}_{NCK}$. We consider the linear map $\prec : \mathbf{M}_{NCK} \otimes \mathbf{M}_{NCK} \to \mathbf{M}_{NCK}$ such that, if $F, G \in \mathbf{M}_{NCK}$ are two nonempty planar forests with $F = F_1 \dots F_n$ and $F_n = B_{NCK}(H)$, then

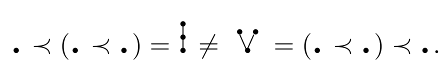
$$F \prec G = F_1 \dots F_{n-1} B_{NCK}(HG).$$

In other terms, G is grafted on the root of the last tree of F, on the right. In particular, we have $\bullet \prec G = B_{NCK}(G)$.

Examples.



Note that $\prec$ is not associative :



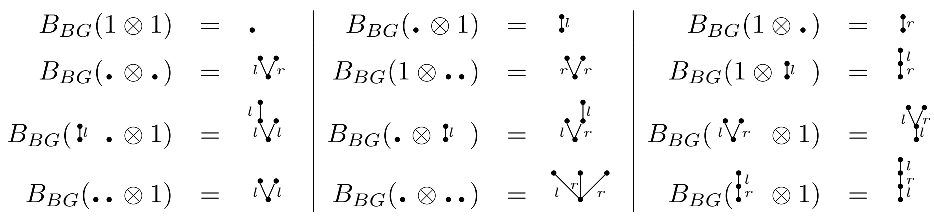
It is proved in [Foi10] that $(\mathbf{M}_{NCK}, m, \prec)$ is the free $\mathcal{RG}$-algebra generated by $\bullet$, where m is the concatenation product.

3.1.3 Free bigraft algebra

Let $\mathcal{D} = \{l, r\}$ be a set of two elements. We denote by $\mathbf{H}^{lr}_{NCK}$ the $\mathbb{K}$-algebra generated by planar forests with their edges decorated by $\mathcal{D}$. Equipped with the product given by the concatenation, $\mathbf{H}^{lr}_{NCK}$ is an algebra.

We define the operator $B_{BG} : \mathbf{H}^{lr}_{NCK} \otimes \mathbf{H}^{lr}_{NCK} \longrightarrow \mathbf{H}^{lr}_{NCK}$, which associates, to a tensor $F \otimes G$ of two forests $F, G \in \mathbf{H}^{lr}_{NCK}$, the tree obtained by grafting the roots of the trees of F and G on a common root and by decorating the edges between the common root and the roots of F by l and the edges between the common root and the roots of G by r.

Examples.

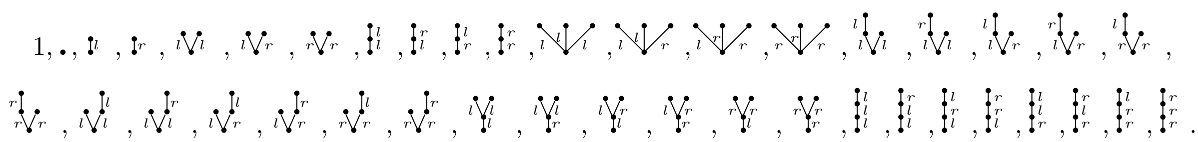


Definition 41 *We define a graded subset $\mathbb{T}_{BG} = \cup_{n\geq 1}\mathbb{T}_{BG}(n)$ of the set of trees of $\mathbf{H}^{lr}_{NCK}$ in the following inductive way :*

- *If $n = 1$, $\mathbb{T}_{BG}(1) = \{\bullet\}$ is the set reduced to one element, the unique tree of degree 1.*
- *If $n \geq 2$, we suppose the set $\bigcup_{1\leq k\leq n-1}\mathbb{T}_{BG}(k)$ already constructed. Pick trees $F_1, \ldots, F_p$ and $G_1, \ldots, G_q \in \bigcup_{1\leq k\leq n-1}\mathbb{T}_{BG}(k)$ such that $|F_1| + \ldots + |F_p| + |G_1| + \ldots + |G_q| = n-1$ and consider the new tree $B_{BG}(F_1 \ldots F_p \otimes G_1 \ldots G_q)$.*

Let $\mathbb{F}_{BG}(n)$ be the forests of degree n constructed with trees of $\mathbb{T}_{BG}$ and $\mathbb{F}_{BG} = \bigcup_{n\geq 0}\mathbb{F}_{BG}(n)$ with $\mathbb{F}_{BG}(0) = \{1\}$.

Examples. Trees of $\mathbb{T}_{BG}$ with degree ≤ 4 :



Lemma 42 *Let $T \in \mathbf{H}^{lr}_{NCK}$ be a nonempty tree. Then $T \in \mathbb{T}_{BG}$ if and only if for all $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$ and w_1 is left to w_2, $(d_1, d_2) \in \{(l,l), (l,r), (r,r)\}$, that is to say that we cannot have $(d_1, d_2) = (r, l)$.*

Proof. By induction on the degree n of T. If $n = 1$, $T = \bullet$ and it is trivial. Let $T \in \mathbf{H}^{lr}_{NCK}$ be a nonempty tree of degree $n \geq 2$ and assume that the property is true at rank $\leq n-1$.

By construction, if $T \in \mathbb{T}_{BG}$, there are $F_1, \ldots, F_p$ and $G_1, \ldots, G_q \in \mathbb{T}_{BG}$ such that $T = B_{BG}(F_1 \ldots F_p \otimes G_1 \ldots G_q)$ and $|F_i|, |G_i| \leq n-1$. Consider $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$ and w_1 is left to w_2. If v is not the root of T, $v \in V(F_i)$ or $v \in V(G_i)$ and by induction hypothesis $(d_1, d_2) \in \{(l,l), (l,r), (r,r)\}$. If v is the root of T, we have three cases :

1. w_1 is the root of F_i and w_2 is the root of F_j with $1 \leq i < j \leq p$. Then $(d_1, d_2) = (l, l)$.
2. w_1 is the root of F_i and w_2 is the root of G_j with $1 \leq i \leq p$ and $1 \leq j \leq q$. Then $(d_1, d_2) = (l, r)$.
3. w_1 is the root of G_i and w_2 is the root of G_j with $1 \leq i < j \leq q$. Then $(d_1, d_2) = (r, r)$.

In all cases, $(d_1, d_2) \in \{(l,l), (l,r), (r,r)\}$.

Reciprocally, let $T \in \mathbf{H}^{lr}_{NCK}$ be a nonempty tree such that for all $v \in V(T)$ and $w_1, w_2 \in V(T)$ such as $w_1 \overset{d_1}{\to} v$, $w_2 \overset{d_2}{\to} v$ and w_1 is left to w_2, $(d_1, d_2) \in \{(l,l), (l,r), (r,r)\}$. If v is the root of T, this condition implies that $T = B_{BG}(F_1 \ldots F_p \otimes G_1 \ldots G_q)$ with $F_i, G_i \in \mathbf{H}^{lr}_{NCK}$ and $|F_i|, |G_i| \leq n-1$. Then F_i and G_i satisfy the condition of the decorations of the edges. Therefore $F_i, G_i \in \mathbb{T}_{BG}$ by induction hypothesis and $T \in \mathbb{T}_{BG}$. $\square$

Let $\mathbf{H}_{BG} = \mathbb{K}(\mathbb{F}_{BG})$ be the subalgebra of $\mathbf{H}^{lr}_{NCK}$ generated by $\mathbb{F}_{BG}$. We denote by $f_n^{\mathbf{H}_{BG}}$ the number of forests of degree n in $\mathbf{H}_{BG}$, $t_n^{\mathbf{H}_{BG}}$ the number of trees of degree n in $\mathbf{H}_{BG}$ and $F_{\mathbf{H}_{BG}}(x) = \sum_{n\geq 0} f_n^{\mathbf{H}_{BG}} x^n$, $T_{\mathbf{H}_{BG}}(x) = \sum_{n\geq 1} t_n^{\mathbf{H}_{BG}} x^n$ the generating series associated. It is possible to calculate $F_{\mathbf{H}_{BG}}(x)$ and $T_{\mathbf{H}_{BG}}(x)$:

Proposition 43 *The generating series $F_{\mathbf{H}_{BG}}$ and $T_{\mathbf{H}_{BG}}$ are given by :*

$$\begin{aligned} F_{\mathbf{H}_{BG}}(x) &= \frac{3}{-1 + 4\cos^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right)}, \\ T_{\mathbf{H}_{BG}}(x) &= \frac{4}{3}\sin^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right). \end{aligned}$$

Proof. $\mathbf{H}_{BG}$ is freely generated by the trees, therefore

$$F_{\mathbf{H}_{BG}}(x) = \frac{1}{1 - T_{\mathbf{H}_{BG}}(x)}. \tag{3.8}$$

We have the following relations :

$$\begin{array}{rcl} t_1^{\mathbf{H}_{BG}} & = & 1, \\ t_n^{\mathbf{H}_{BG}} & = & \displaystyle\sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n-1} (k+1) t_{a_1}^{\mathbf{H}_{BG}} \ldots t_{a_k}^{\mathbf{H}_{BG}} \text{ if } n \geq 2. \end{array}$$

Then

$$T_{\mathbf{H}_{BG}}(x) - x = \sum_{k=1}^{\infty} (k+1) x T_{\mathbf{H}_{BG}}(x)^k = xF \circ T_{\mathbf{H}_{BG}}(x),$$

where $F(h) = \displaystyle\sum_{k=1}^{\infty}(k+1)h^k = \frac{2h-h^2}{(1-h)^2}$. We deduce the following equality :

$$T_{\mathbf{H}_{BG}}(x)^3 - 2T_{\mathbf{H}_{BG}}(x)^2 + T_{\mathbf{H}_{BG}}(x) = x. \tag{3.9}$$

So $T_{\mathbf{H}_{BG}}(x)$ is the inverse for the composition of $x^3 - 2x^2 + x$, that is to say

$$T_{\mathbf{H}_{BG}}(x) = \frac{4}{3}\sin^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right).$$

Remark that, with (3.9), we obtain an inductive definition of the coefficients $t_n^{\mathbf{H}_{BG}}$:

$$\left\{ \begin{array}{l} t_1^{\mathbf{H}_{BG}} = 1, \\ t_n^{\mathbf{H}_{BG}} = 2 \displaystyle\sum_{i,j\geq 1 \text{ and } i+j=n} t_i^{\mathbf{H}_{BG}} t_j^{\mathbf{H}_{BG}} - \sum_{i,j,k\geq 1 \text{ and } i+j+k=n} t_i^{\mathbf{H}_{BG}} t_j^{\mathbf{H}_{BG}} t_k^{\mathbf{H}_{BG}} \text{ if } n \geq 2. \end{array} \right.$$

We deduce from (3.8) the equality

$$F_{\mathbf{H}_{BG}}(x) = \frac{3}{-1 + 4\cos^2\left(\frac{1}{3}\arcsin\left(\sqrt{\frac{27x}{4}}\right)\right)}.$$

□

This gives :

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{H}_{BG}}$	1	2	7	30	143	728	3876	21318	120175	690690
$f_n^{\mathbf{H}_{BG}}$	1	3	12	55	273	1428	7752	43263	246675	1430715

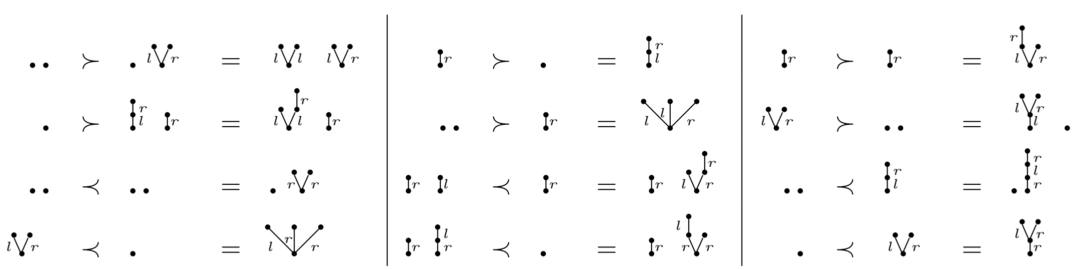
These are the sequences A006013 and A001764 in [Slo].

Definition 44 *Let $F, G \in \mathbb{F}_{BG} \setminus \{1\}$. Suppose that $F = F_1 \ldots F_n$, $G = G_1 \ldots G_m$, with $F_i, G_i \in \mathbb{T}_{BG}$, $F_1 = B_{BG}(F_1^1 \otimes F_1^2)$ and $G_m = B_{BG}(G_m^1 \otimes G_m^2)$. We define :*

$$\begin{array}{rcl} G \succ F & = & B_{BG}(GF_1^1 \otimes F_1^2) F_2 \ldots F_n, \\ G \prec F & = & G_1 \ldots G_{m-1} B_{BG}(G_m^1 \otimes G_m^2 F). \end{array}$$

We denote by $\mathbf{M}_{BG}$ the algebra generated by the nonempty forests of $\mathbf{H}_{BG}$. Its product is given by the concatenation of forests. We extend $\succ, \prec : \mathbf{M}_{BG} \otimes \mathbf{M}_{BG} \to \mathbf{M}_{BG}$ by linearity.

Examples.



In other terms, for $G \succ F$, G is grafted on the root of the first tree of F on the left with edges decorated by l. For $G \prec F$, F is grafted on the root of the last tree of G on the right with edges decorated by r. In particular, $F \succ \bullet = B_{BG}(F \otimes 1)$ and $\bullet \prec F = B_{BG}(1 \otimes F)$.

Remark. $\succ$ and $\prec$ are not associative :

$\bullet \succ (\bullet \succ \bullet) =$  $\neq (\bullet \succ \bullet) \succ \bullet =$  ,

$\bullet \prec (\bullet \prec \bullet) =$  $\neq (\bullet \prec \bullet) \prec \bullet =$  .

Proposition 45 $\mathbf{M}_{BG}$ *is given a graded $\mathcal{BG}$-algebra structure : for all $x, y, z \in \mathbf{M}_{BG}$:*

$$\begin{aligned}
(xy) \succ z &= x \succ (y \succ z), & (3.10)\\
(x \succ y)z &= x \succ (yz), & (3.11)\\
(x \prec y) \prec z &= x \prec (yz), & (3.12)\\
(xy) \prec z &= x(y \prec z), & (3.13)\\
(x \succ y) \prec z &= x \succ (y \prec z). & (3.14)
\end{aligned}$$

Proof. We can restrict ourselves to $x, y, z \in \mathbb{F}_{BG} \setminus \{1\}$. (3.11) and (3.13) are immediate. We put $x = x_1 B_{BG}(x_2 \otimes x_3)$ and $z = B_{BG}(z_1 \otimes z_2)z_3$ with $x_i, z_i \in \mathbb{F}_{BG}$. Then

$$x \succ (y \succ z) = x \succ (B_{BG}(yz_1 \otimes z_2)z_3) = B_{BG}(xyz_1 \otimes z_2)z_3 = (xy) \succ (B_{BG}(z_1 \otimes z_2)z_3) = (xy) \succ z,$$
$$(x \prec y) \prec z = (x_1 B_{BG}(x_2 \otimes x_3 y)) \prec z = x_1 B_{BG}(x_2 \otimes x_3 yz) = (x_1 B_{BG}(x_2 \otimes x_3)) \prec (yz) = x \prec (yz).$$

So we have proved (3.10) and (3.12). In order to prove (3.14), we must study two cases :

1. If $y = B_{BG}(y_1 \otimes y_2)$ is a tree,
$$(x \succ y) \prec z = B_{BG}(xy_1 \otimes y_2) \prec z = B_{BG}(xy_1 \otimes y_2 z) = x \succ B_{BG}(y_1 \otimes y_2 z) = x \succ (y \prec z).$$
2. If $y = y_1 y_2$ with $y_1, y_2 \in \mathbb{F}_{BG} \setminus \{1\}$,
$$(x \succ y) \prec z = ((x \succ y_1)y_2) \prec z = (x \succ y_1)(y_2 \prec z) = x \succ (y_1(y_2 \prec z)) = x \succ (y \prec z),$$
by (3.11) and (3.13).

$\square$

We now prove that $\mathbf{M}_{BG}$ is the free $\mathcal{BG}$-algebra generated by $\bullet$ and deduce the dimension of $\mathcal{BG}(n)$ for all n.

Theorem 46 $(\mathbf{M}_{BG}, m, \succ, \prec)$ *is the free $\mathcal{BG}$-algebra generated by $\bullet$.*

Proof. Let A be a $\mathcal{BG}$-algebra and let $a \in A$. Let us prove that there exists a unique morphism of $\mathcal{BG}$-algebras $\phi : \mathbf{M}_{BG} \to A$, such that $\phi(\bullet) = a$. We define $\phi(F)$ for any nonempty forest $F \in \mathbf{M}_{BG}$ inductively on the degree of F by :

$$\left\{\begin{array}{rcl}
\phi(\bullet) &=& a,\\
\phi(F_1 \ldots F_k) &=& \phi(F_1) \ldots \phi(F_k) \text{ if } k \geq 2,\\
\phi(F) &=& \big(\phi(F^1) \succ a\big) \prec \phi(F^2) \text{ if } F = B_{BG}(F^1 \otimes F^2) \text{ with } F^1, F^2 \in \mathbb{F}_{BG}.
\end{array}\right.$$

As the product of A is associative, this is perfectly defined. This map is linearly extended into a map $\phi : \mathbf{M}_{BG} \to A$. Let us show that it is a morphism of $\mathcal{BG}$-algebras. By the second point, $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in \mathbf{M}_{BG}$. Let F, G be two nonempty trees. Let us prove that $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$. Denote $F = B_{BG}(F^1 \otimes F^2)$, $G = B_{BG}(G^1 \otimes G^2)$ with F^1, F^2 and G^1, G^2 in $\mathbb{F}_{BG}$. Then :

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,
$$\begin{aligned}
\phi(F \succ G) &= \phi(B_{BG}(FG^1 \otimes G^2))\\
&= \big(\phi(FG^1) \succ a\big) \prec \phi(G^2)\\
&= \big(\phi(F) \succ \big(\phi(G^1) \succ a\big)\big) \prec \phi(G^2)\\
&= \phi(F) \succ \big(\big(\phi(G^1) \succ a\big) \prec \phi(G^2)\big)\\
&= \phi(F) \succ \phi(G).
\end{aligned}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{array}{rcl}\phi(F \prec G) &=& \phi(B_{BG}(F^1 \otimes F^2 G)) \\ &=& \left(\phi(F^1) \succ a\right) \prec \phi(F^2 G) \\ &=& \phi(F^1) \succ \left(a \prec \phi(F^2 G)\right) \\ &=& \phi(F^1) \succ \left(\left(a \prec \phi(F^2)\right) \prec \phi(G)\right) \\ &=& \left(\left(\phi(F^1) \succ a\right) \prec \phi(F^2)\right) \prec \phi(G) \\ &=& \phi(F) \prec \phi(G).\end{array}$$

If F, G are two nonempty forests, $F = F_1 \dots F_n$, $G = G_1 \dots G_m$. Then :

$$\begin{array}{rcl}\phi(F \succ G) &=& \phi((F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1)\dots))G_2 \dots G_m) \\ &=& \phi(F_1 \succ (\dots F_{n-1} \succ (F_n \succ G_1)\dots))\phi(G_2 \dots G_m) \\ &=& (\phi(F_1) \succ (\dots \phi(F_{n-1}) \succ (\phi(F_n) \succ \phi(G_1))\dots))\phi(G_2 \dots G_m) \\ &=& (\phi(F_1 \dots F_n) \succ \phi(G_1))\phi(G_2 \dots G_m) \\ &=& \phi(F) \succ \phi(G),\end{array}$$

and

$$\begin{array}{rcl}\phi(F \prec G) &=& \phi(F_1 \dots F_{n-1}(\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &=& \phi(F_1 \dots F_{n-1})\phi((\dots (F_n \prec G_1) \prec G_2 \dots) \prec G_m) \\ &=& \phi(F_1 \dots F_{n-1})((\dots (\phi(F_n) \prec \phi(G_1)) \prec \phi(G_2)\dots) \prec \phi(G_m)) \\ &=& \phi(F_1 \dots F_{n-1})(\phi(F_n) \prec \phi(G_1 \dots G_m)) \\ &=& \phi(F) \prec \phi(G).\end{array}$$

So ϕ is a morphism of $\mathcal{BG}$-algebras.

Let $\phi' : \mathbf{M}_{BG} \to A$ be another morphism of $\mathcal{BG}$-algebras such that $\phi'(\bullet) = a$. Then for any planar trees $F_1, \dots, F_k$, $\phi'(F_1 \dots F_k) = \phi'(F_1)\dots\phi'(F_k)$. For any forests $F^1, F^2 \in \mathbf{M}_{BG}$,

$$\begin{array}{rcl}\phi'(B_{BG}(F^1 \otimes F^2)) &=& \phi'((F^1 \succ \bullet) \prec F^2) \\ &=& (\phi'(F^1) \succ \phi'(\bullet)) \prec \phi'(F^2) \\ &=& (\phi'(F^1) \succ a) \prec \phi'(F^2).\end{array}$$

So $\phi = \phi'$. □

Proposition 47 *Let $\dagger : \mathbf{H}_{BG} \to \mathbf{H}_{BG}$ be the $\mathbb{K}$-linear map built by induction as follows : $1^\dagger = 1$ and for all $F, G \in \mathbb{F}_{BG}$, $(FG)^\dagger = G^\dagger F^\dagger$ and $(B_{BG}(F \otimes G))^\dagger = B_{BG}(G^\dagger \otimes F^\dagger)$. Then $\dagger$ is an involution over $\mathbf{H}_{BG}$.*

Proof. Let us prove that $(F^\dagger)^\dagger = F$ for all $F \in \mathbb{F}_{BG}$ by induction on the degree n of F. If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B_{BG}(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}_{BG}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (B_{BG}(H^\dagger \otimes G^\dagger))^\dagger = B_{BG}((G^\dagger)^\dagger \otimes (H^\dagger)^\dagger) = B_{BG}(G \otimes H) = F$, using the induction hypothesis for G and H.
2. If $F = GH$ is a forest, with $G, H \in \mathbb{F}_{BG} \setminus \{1\}$ such that $|G|, |H| < n$. Then $(F^\dagger)^\dagger = (H^\dagger G^\dagger)^\dagger = (G^\dagger)^\dagger (H^\dagger)^\dagger = GH = F$, using again the induction hypothesis for G and H.

In all cases, $(F^\dagger)^\dagger = F$. □

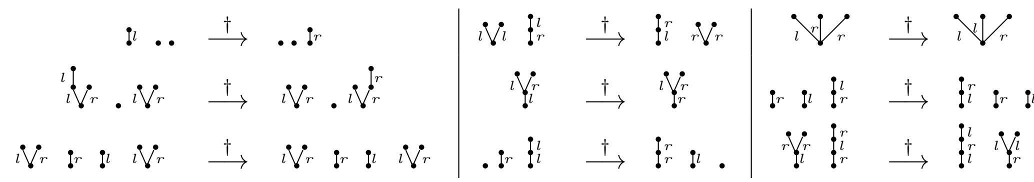
Remark.

1. Let $F \in \mathbf{H}_{BG}$. The forest $F^\dagger$ is obtaiend by inverting the total orders on the set of roots of F and the sets $\{w \in V(F) \mid w \to v\}$ for any $v \in V(F)$ and by exchanging the decorations l and r.
2. We can rewrite the relations between $\dagger$ and the product m and B_{BG} as follows :

$$\begin{array}{rcl}m \circ (\dagger \otimes \dagger) &=& \dagger \circ m \circ \tau, \\ B_{BG} \circ (\dagger \otimes \dagger) &=& \dagger \circ B_{BG} \circ \tau.\end{array}$$

where τ is the flip defined in the notations.

Examples.



The involution $\dagger : \mathbf{H}_{BG} \to \mathbf{H}_{BG}$ permits to exchange the structures of right and left graft algebra on $\mathbf{H}_{BG}$:

Proposition 48 *For all forests $F, G \in \mathbf{M}_{BG}$, $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ and $(G \prec F)^\dagger = F^\dagger \succ G^\dagger$.*

Proof. Let $F, G \in \mathbf{M}_{BG}$ be two forests. We put $F = F_1 \dots F_n$, $G = G_1 \dots G_m$, $F_1 = B_{BG}(F_1^1 \otimes F_1^2)$ and $G_m = B_{BG}(G_m^1 \otimes G_m^2)$. Then

$$\begin{array}{rcl} (G \succ F)^\dagger & = & (B_{BG}(GF_1^1 \otimes F_1^2)F_2 \dots F_n)^\dagger \\ & = & F_n^\dagger \dots F_2^\dagger B_{BG}((F_1^2)^\dagger \otimes (GF_1^1)^\dagger) \\ & = & F_n^\dagger \dots F_2^\dagger B_{BG}((F_1^2)^\dagger \otimes (F_1^1)^\dagger G^\dagger) \\ & = & \left(F_n^\dagger \dots F_2^\dagger B_{BG}((F_1^2)^\dagger \otimes (F_1^1)^\dagger)\right) \prec G^\dagger \\ & = & F^\dagger \prec G^\dagger . \end{array}$$

Therefore $(G \succ F)^\dagger = F^\dagger \prec G^\dagger$ for all forests $F, G \in \mathbf{M}_{BG}$. Moreover, as $\dagger$ is an involution over $\mathbf{M}_{BG}$ (proposition 47), $(G \prec F)^\dagger = ((G^\dagger)^\dagger \prec (F^\dagger)^\dagger)^\dagger = ((F^\dagger \succ G^\dagger)^\dagger)^\dagger = F^\dagger \succ G^\dagger$. □

3.1.4 The bigraft operad

Recall that $\widetilde{\mathcal{BG}}$ is the nonsymmetric operad associated with the regular operad $\mathcal{BG}$. With the following proposition, we can identify $\mathcal{BG}$ with the vector space of nonempty forests $\in \mathbf{H}_{BG}$.

Proposition 49 *For all $n \in \mathbb{N}^*$, $dim(\widetilde{\mathcal{BG}}(n)) = f_n^{\mathbf{H}_{BG}}$ and the following map is bijective :*

$$\Psi : \left\{ \begin{array}{rcl} \widetilde{\mathcal{BG}}(n) & \to & Vect(forests \in \mathbf{H}_{BG} \text{ of degree } n) \subseteq \mathbf{M}_{BG} \\ p & \to & p.(\bullet, \dots, \bullet). \end{array} \right. \tag{3.15}$$

Proof. It suffices to show that Ψ is bijective. $(\mathbf{M}_{BG}, m, \succ, \prec)$ is generated by $\bullet$ as $\mathcal{BG}$-algebra (with theorem 46) therefore Ψ is surjective. Ψ is injective by the freedom in theorem 46. □

In the remainder of this section, we identify $F \in \mathbb{F}_{BG}(n)$ and $\Psi^{-1}(F) \in \mathcal{BG}(n)$.

Notations. In order to distinguish the composition in $\mathcal{BG}$ and the action of the operad $\mathcal{BG}$ on $\mathbf{M}_{BG}$, we now denote by

1. $F \circ (F_1, \dots, F_n)$ the composition of $\mathcal{BG}$.
2. $F \bullet (F_1, \dots, F_n)$ the action of $\mathcal{BG}$ on $\mathbf{M}_{BG}$.

In the following theorem, we describe the composition of $\mathcal{BG}$ in term of forests.

Theorem 50 *The composition of $\mathcal{BG}$ in the basis of forests belonging to $\mathbb{F}_{BG} \setminus \{1\}$ can be inductively defined in this way :*

$$\begin{array}{rcl} \bullet \circ (H) & = & H, \\ FG \circ (H_1, \dots, H_{|F|+|G|}) & = & F \circ (H_1, \dots, H_{|F|})G \circ (H_{|F|+1}, \dots, H_{|F|+|G|}), \\ B_{BG}(F \otimes G) \circ (H_1, \dots, H_{|F|+|G|+1}) & = & \big((F \circ (H_1, \dots, H_{|F|})) \succ H_{|F|+1}\big) \\ & & \prec (G \circ (H_{|F|+2}, \dots, H_{|F|+|G|+1})). \end{array}$$

Proof. Note that $\Psi(\cdot) = \cdot = \Psi(I)$. Hence, $\cdot$ is the unit element of $\mathcal{BG}$.

By definition, $\Psi(\cdot\,\cdot) = \cdot\,\cdot = \Psi(m)$. So $\cdot\,\cdot = m$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F}_{BG} \setminus \{1\}$,

$$\begin{array}{rcl} \Psi(FG) & = & FG \\ & = & m \bullet (F, G) \\ & = & m \bullet (F \bullet (\cdot, \ldots, \cdot), G \bullet (\cdot, \ldots, \cdot)) \\ & = & (m \circ (F, G)) \bullet (\cdot, \ldots, \cdot) \\ & = & \Psi(m \circ (F, G)). \end{array}$$

So $FG = m \circ (F, G) = \cdot\,\cdot \circ (F, G)$.

We have $\Psi(\vdots^{l}) = \cdot \succ \cdot = \Psi(\succ)$. So $\vdots^{l} = \succ$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F}_{BG} \setminus \{1\}$,

$$\begin{array}{rcl} \Psi(F \succ G) & = & F \succ G \\ & = & \succ \bullet(F, G) \\ & = & \succ \bullet(F \bullet (\cdot, \ldots, \cdot), G \bullet (\cdot, \ldots, \cdot)) \\ & = & (\succ \circ(F, G)) \bullet (\cdot, \ldots \cdot) \\ & = & \Psi(\succ \circ(F, G)). \end{array}$$

So $F \succ G = \succ \circ(F, G) = \vdots^{l} \circ (F, G)$.

As $\Psi(\vdots^{r}) = \cdot \prec \cdot = \Psi(\prec)$, $\vdots^{r} = \prec$ in $\mathcal{BG}(2)$. Moreover, for all $F, G \in \mathbb{F}_{BG} \setminus \{1\}$,

$$\begin{array}{rcl} \Psi(F \prec G) & = & F \prec G \\ & = & \prec \bullet(F, G) \\ & = & \prec \bullet(F \bullet (\cdot, \ldots, \cdot), G \bullet (\cdot, \ldots, \cdot)) \\ & = & (\prec \circ(F, G)) \bullet (\cdot, \ldots \cdot) \\ & = & \Psi(\prec \circ(F, G)). \end{array}$$

So $F \prec G = \prec \circ(F, G) = \vdots^{r} \circ (F, G)$.

Let $F \in \mathbb{F}_{BG}(m)$ and $G \in \mathbb{F}_{BG}(n)$ with $m, n \geq 1$. Let $H_1, \ldots, H_{m+n+1} \in \mathbb{F}_{BG} \setminus \{1\}$. We will show that, in $\mathcal{BG}$,

$$\begin{array}{rcl} (FG) \circ (H_1, \ldots, H_{m+n}) & = & F \circ (H_1, \ldots, H_m) G \circ (H_{m+1}, \ldots, H_{m+n}) \\ B_{BG}(F \otimes G) \circ (H_1, \ldots, H_{m+n+1}) & = & ((F \circ (H_1, \ldots, H_m)) \succ H_{m+1}) \\ & & \prec (G \circ (H_{m+2}, \ldots, H_{m+n+1})). \end{array}$$

Indeed, in $\mathcal{BG}$,

$$\begin{array}{rcl} (FG) \circ (H_1, \ldots, H_{m+n}) & = & (m \circ (F, G)) \circ (H_1, \ldots, H_{m+n}) \\ & = & m \circ (F \circ (H_1, \ldots, H_m), G \circ (H_{m+1}, \ldots, H_{m+n})) \\ & = & F \circ (H_1, \ldots, H_m) G \circ (H_{m+1}, \ldots, H_{m+n}), \end{array}$$

and

$$\begin{array}{rl} & B_{BG}(F \otimes G) \circ (H_1, \ldots, H_{m+n+1}) \\ = & ((F \succ \cdot) \prec G) \circ (H_1, \ldots, H_{m+n+1}) \\ = & (\vdots^{r} \circ (\vdots^{l} \circ (F, \cdot), G)) \circ (H_1, \ldots, H_{m+n+1}) \\ = & \vdots^{r} \circ ((\vdots^{l} \circ (F, \cdot)) \circ (H_1, \ldots, H_{m+1}), G \circ (H_{m+2}, \ldots, H_{m+n+1})) \\ = & \vdots^{r} \circ (\vdots^{l} \circ (F \circ (H_1, \ldots, H_m), \cdot \circ H_{m+1}), G \circ (H_{m+2}, \ldots, H_{m+n+1})) \\ = & \vdots^{r} \circ ((F \circ (H_1, \ldots, H_m)) \succ H_{m+1}, G \circ (H_{m+2}, \ldots, H_{m+n+1})) \\ = & ((F \circ (H_1, \ldots, H_m)) \succ H_{m+1}) \prec (G \circ (H_{m+2}, \ldots, H_{m+n+1})). \end{array}$$

$\square$

Examples. Let $F_1, F_2, F_3 \in \mathbb{F}_{BG} \setminus \{1\}$.

 $\circ (F_1, F_2) = F_1F_2$

 $\circ (F_1, F_2) = F_1 \succ F_2$

 $\circ (F_1, F_2) = F_1 \prec F_2$

 $\circ (F_1, F_2, F_3) = F_1(F_2 \succ F_3)$

 $\circ (F_1, F_2, F_3) = (F_1 \prec F_2)F_3$

 $\circ (F_1, F_2, F_3) = F_1F_2F_3$

 $\circ (F_1, F_2, F_3) = (F_1 \succ F_2) \prec F_3$

 $\circ (F_1, F_2, F_3) = F_1 \prec (F_2F_3)$

 $\circ (F_1, F_2, F_3) = (F_1F_2) \succ F_3$

 $\circ (F_1, F_2, F_3) = (F_1 \prec F_2) \succ F_3$

 $\circ (F_1, F_2, F_3) = F_1 \prec (F_2 \succ F_3)$

 $\circ (F_1, F_2, F_3) = (F_1 \succ F_2) \succ F_3$

The free $\mathcal{BG}$-algebra over a vector space V is the $\mathcal{BG}$-algebra $\mathcal{BG}(V)$ such that any map from V to a $\mathcal{BG}$-algebra A has a natural extention as a $\mathcal{BG}$-morphism $\mathcal{BG}(V) \to A$. In other words the functor $\mathcal{BG}(-)$ is the left adjoint to the forgetful functor from $\mathcal{BG}$-algebras to vector spaces. Because the operad $\mathcal{BG}$ is regular, we get the following result :

Proposition 51 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{BG}$-algebra on V is*

$$\mathcal{BG}(V) = \bigoplus_{n\geq 1} \mathbb{K}\left(\mathbb{F}_{BG}(n)\right) \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{F}_{BG}(n)$, $G \in \mathbb{F}_{BG}(m)$, $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \ldots \otimes w_m \in V^{\otimes m}$,

$$\begin{aligned}
(F \otimes v_1 \otimes \ldots \otimes v_n) * (G \otimes w_1 \otimes \ldots \otimes w_m) &= (FG \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m),\\
(F \otimes v_1 \otimes \ldots \otimes v_n) \succ (G \otimes w_1 \otimes \ldots \otimes w_m) &= (F \succ G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m),\\
(F \otimes v_1 \otimes \ldots \otimes v_n) \prec (G \otimes w_1 \otimes \ldots \otimes w_m) &= (F \prec G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m).
\end{aligned}$$

3.2 Koszul duality

3.2.1 Dual operad of the bigraft operad

As $\mathcal{BG}$ is a binary and quadratic operad, it admits a dual, in the sense of V. Ginzburg and M. Kapranov (see [GK94, MSS02]). We denote by $\mathcal{BG}^!$ the dual operad of $\mathcal{BG}$. We prove that $\mathcal{BG}^!$ is defined as follows :

Lemma 52 *The operad $\mathcal{BG}^!$, dual of $\mathcal{BG}$, is the quadratic operad generated by three operations m, $\succ$ and $\prec$ and satisfying the following relations, where I is the unit element of $\mathcal{BG}^!$:*

$$\left\{\begin{array}{l}
r_1 = \succ \circ(m, I) - \succ \circ(I, \succ),\\
r_2 = m \circ (\succ, I) - \succ \circ(I, m),\\
r_3 = \prec \circ(\prec, I) - \prec \circ(I, m),\\
r_4 = \prec \circ(m, I) - m \circ (I, \prec),\\
r_5 = \prec \circ(\succ, I) - \succ \circ(I, \prec),\\
r_6 = m \circ (m, I) - m \circ (I, m),
\end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{l}
r_7 = \succ \circ(\succ, I),\\
r_8 = \succ \circ(\prec, I),\\
r_9 = m \circ (\prec, I),\\
r_{10} = \prec \circ(I, \prec),\\
r_{11} = \prec \circ(I, \succ),\\
r_{12} = m \circ (I, \succ).
\end{array}\right. \tag{3.16}$$

The notations r_1 to r_{12} of the relations are introduced for future references.

Proof. Recall some notations. For any right Σ_n-module V, we denote by $V^!$ the right Σ_n-module $V^* \otimes (\varepsilon)$, where (ε) is the one-dimensional signature representation. Explicitly, the action of Σ_n on V^* is defined by $f^\sigma(x) = \varepsilon(\sigma)f(x^{\sigma^{-1}})$ for all $\sigma \in \Sigma_n$, $f \in V^!$, $x \in V$. The pairing between $V^!$ and V is given by : if $f \in V^!$, $x \in V$,

$$\langle -, - \rangle : V^! \otimes V \longrightarrow \mathbb{K}, \langle f, x\rangle = f(x), \tag{3.17}$$

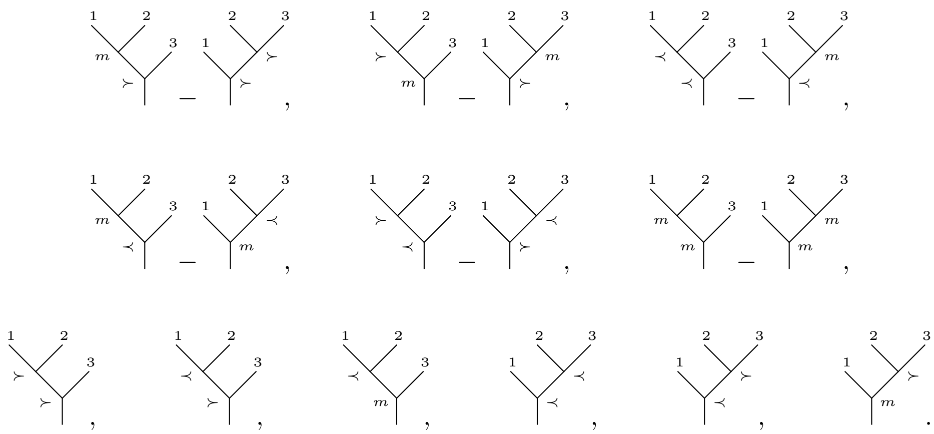
and if $\sigma \in \Sigma_n$, $\langle f^\sigma, x^\sigma\rangle = \varepsilon(\sigma)\langle f, x\rangle$.

The operad $\mathcal{BG}$ is generated by the free Σ_2-module E, generated by m, $\succ$ and $\prec$, with relations $R \subseteq \mathcal{P}_E(3)$, where $\mathcal{P}_E(3)$ is the free operad generated by E and R is the free Σ_3-submodule of $\mathcal{P}_E(3)$ generated by $r_1, \ldots, r_6$. Note that $dim(E) = 6$, $dim(\mathcal{P}_E(3)) = 108$ and $dim(R) = 36$. So $\mathcal{BG}^!$ is generated by the dual $E^!$, with relations $R^\perp$ the annihilator of R for the pairing (3.17). So $dim(R^\perp) =$

$dim(\mathcal{P}_E(3)) - dim(R) = 108 - 36 = 72$. We then verify that the given relations (3.16) for $\mathcal{BG}^!$ are indeed in $R^\perp$, that each of them generates a free Σ_3-module, and finally that there Σ_3-modules are in direct sum. So these relations entirely generate $R^\perp$. $\square$

Remarks.

1. $\mathcal{BG}^!$ is a quotient of $\mathcal{BG}$.
2. $\mathcal{BG}^!$ is the symmetrization of the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$ generated by $m, \succ$ and $\prec$ and satisfying the relations r_1 to r_{12}.
3. Graphically, the relations defining $\mathcal{BG}^!$ can be written in the following way :



We now give a combinatorial description of the free $\mathcal{BG}^!$-algebra. We will show that it is a quotient of the free $\mathcal{BG}$-algebra $\mathbf{H}_{BG}$.

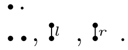
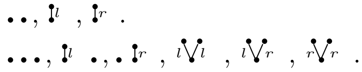
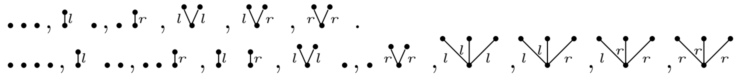
Definition 53 *Let $\mathbb{F}^!_{BG}$ be the subset of forests of $\mathbb{F}_{BG}$ containing the empty tree 1 and satisfying the following conditions : if $F_1 \dots F_n \in \mathbb{F}^!_{BG}$ with the F_i's nonempty trees, then*

1. *$F_1, \dots, F_n$ are corollas $\in \mathbb{F}_{BG}$,*
2. *if $\exists\ e \in E(F_i)$ decorated by l, then $i = 1$,*
3. *if $\exists\ e \in E(F_i)$ decorated by r, then $i = n$.*

We set $\mathbb{G}_{BG} = \mathbb{F}_{BG} \setminus \mathbb{F}^!_{BG}$.

Remark. If $F_1 \dots F_n \in \mathbb{F}^!_{BG}$ with $n \geq 2$, then $F_2, \dots, F_{n-1} = \bullet$, every $e \in E(F_1)$ is decorated by l and every $e \in E(F_n)$ is decorated by r. In particular, we have $\dagger(\mathbb{F}^!_{BG}) = \mathbb{F}^!_{BG}$, where $\dagger : \mathbf{H}_{BG} \to \mathbf{H}_{BG}$ is the involution defined in the proposition 47.

Examples. Forests of $\mathbb{F}^!_{BG}$:

– In degree 1 : $\bullet$.
– In degree 2 :  .
– In degree 3 :  .
– In degree 4 :  .

Let $\mathbf{H}^!_{BG}$ be the $\mathbb{K}$-vector space spanned by $\mathbb{F}^!_{BG}$ and $\mathbf{M}^!_{BG}$ the $\mathbb{K}$-vector space spanned by $\mathbb{F}^!_{BG} \setminus \{1\}$. We denote by $t_n^{\mathbf{H}^!_{BG}}$ the number of trees of degree n in $\mathbf{H}^!_{BG}$ and $f_n^{\mathbf{H}^!_{BG}}$ the number of forests of degree n in $\mathbf{H}^!_{BG}$. We put $T_{\mathbf{H}^!_{BG}}(x) = \sum_{n\geq 1} t_n^{\mathbf{H}^!_{BG}} x^n$ and $F_{\mathbf{H}^!_{BG}}(x) = \sum_{n\geq 1} f_n^{\mathbf{H}^!_{BG}} x^n$.

Proposition 54 *The generating series of $\mathbf{H}^!_{BG}$ are given by :*

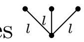
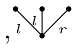
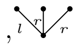
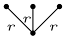
$$T_{\mathbf{H}^!_{BG}}(x) = \frac{x}{(1-x)^2} \qquad \text{and} \qquad F_{\mathbf{H}^!_{BG}}(x) = \frac{x}{(1-x)^3}.$$

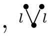
Proof. We have $t_1^{\mathbf{H}^!_{BG}} = 1$, $f_1^{\mathbf{H}^!_{BG}} = 1$. For all $n \geq 2$, $t_n^{\mathbf{H}^!_{BG}} = n$ because the corollas of degree n have $n-1$ edges with n possible different decorations. Moreover, for all $n \geq 2$,

$$f_n^{\mathbf{H}^!_{BG}} = f_{n-1}^{\mathbf{H}^!_{BG}} - t_{n-1}^{\mathbf{H}^!_{BG}} + t_n^{\mathbf{H}^!_{BG}} + (n-1)$$

where the term :

– $f_{n-1}^{\mathbf{H}^!_{BG}} - t_{n-1}^{\mathbf{H}^!_{BG}}$ corresponds to forests of degree n and of length ≥ 3 obtained from the forests of degree $n-1$ and of length ≥ 2 by adding $\bullet$ in the middle. For example, in degree 4, these are the forests $\bullet\bullet\bullet\bullet$, $\bullet\bullet$ and $\bullet\bullet$  .

– $t_n^{\mathbf{H}^!_{BG}}$ corresponds to trees of degree n. In degree 4, these are the trees  ,  ,  and  .

– $n-1$ corresponds to forests of degree n and of length 2 obtained by product of a tree of degree k and a tree of degree $n-k$, $1 \leq k \leq n-1$. For example, in degree 4, these are the forests   ,  $\bullet$ and $\bullet$  .

So $f_n^{\mathbf{H}^!_{BG}} = f_{n-1}^{\mathbf{H}^!_{BG}} + n = \frac{n(n+1)}{2}$. We deduce that $T_{\mathbf{H}^!_{BG}}(x) = \frac{x}{(1-x)^2}$ and $F_{\mathbf{H}^!_{BG}}(x) = \frac{x}{(1-x)^3}$. $\square$

Proposition 55 *$(\mathbf{M}^!_{BG}, m, \succ, \prec)$ is a $\mathcal{BG}^!$-algebra generated by $\bullet$.*

Proof. Let us prove that if $F, G \in \mathbb{F}_{BG}$ are two nonempty forests such that $F \in \mathbb{G}_{BG}$ or $G \in \mathbb{G}_{BG}$, then $FG, F \succ G$ and $F \prec G \in \mathbb{G}_{BG}$.

Suppose that $F \notin \mathbb{F}^!_{BG}$. We have two cases :

1. If F is not a monomial of corollas. Then $h(F) \geq 2$. So $h(F \bullet G), h(G \bullet F) \geq 2$ and $F \bullet G \in \mathbb{G}_{BG}, G \bullet F \in \mathbb{G}_{BG}$ for all $\bullet \in \{m, \succ, \prec\}$.
2. If F is a monomial of corollas. As $F \notin \mathbb{F}^!_{BG}$, $h(F) \geq 1$ and $F = F_1F_2$ with F_1, F_2 nonempty such that $\exists\ e \in E(F_2)$ decorated by l (this is the same argument with r). Then $FG, GF \in \mathbb{G}_{BG}$, $G \succ F = (G \succ F_1)F_2 \in \mathbb{G}_{BG}$ and $F \prec G = F_1(F_2 \prec G) \in \mathbb{G}_{BG}$. Moreover $h(F \succ G), h(G \prec F) \geq 2$ and $F \succ G, G \prec F \in \mathbb{G}_{BG}$.

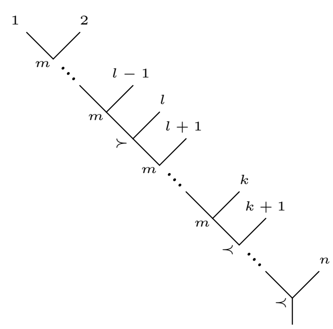
So $\mathbb{K}(\mathbb{G}_{BG})$ is a $\mathcal{BG}$-ideal of the $\mathcal{BG}$-algebra $\mathbf{M}_{BG}$ and the quotient vector space $\mathbf{M}^!_{BG} = \mathbf{M}_{BG}/\mathbb{K}(\mathbb{G}_{BG})$ is a $\mathcal{BG}$-algebra. In the sequel, we shall identify a forest $F \in \mathbf{M}_{BG}$ and its class in $\mathbf{M}^!_{BG}$.

It remains to show relations $r7$ to $r12$. Let $F, G, H \in \mathbf{M}_{BG}$ be three nonempty forests. $h((F \succ G) \succ H), h((F \prec G) \succ H), h(F \prec (G \prec H))$ and $h(F \prec (G \succ H)) \geq 3$. Therefore $(F \succ G) \succ H, (F \prec G) \succ H, F \prec (G \prec H), F \prec (G \succ H) \in \mathbb{G}_{BG}$ and the relations $r7, r8, r10, r11$ are satisfied in $\mathbf{M}^!_{BG}$. Moreover $(F \prec G)H, F(G \succ H) \in \mathbb{G}_{BG}$ by considering the decorations of the edges and $r9, r12$ are true in $\mathbf{M}^!_{BG}$. So $(\mathbf{M}^!_{BG}, m, \succ, \prec)$ is a $\mathcal{BG}^!_{BG}$-algebra. As $\mathbf{M}_{BG}$ is generated as $\mathcal{BG}$-algebra by $\bullet$ (with theorem 46), $\mathbf{M}^!_{BG}$ is also generated by $\bullet$ as $\mathcal{BG}$-algebra and therefore as $\mathcal{BG}^!$-algebra. $\square$

Theorem 56 *1. $(\mathbf{M}^!_{BG}, m, \succ, \prec)$ is the free $\mathcal{BG}^!$-algebra generated by $\bullet$.*

2. For all $n \in \mathbb{N}^*$, $\mathcal{BG}^!(n) = \frac{n(n+1)!}{2}$.

3. For all $n \in \mathbb{N}^*$, $\mathcal{BG}^!(n)$ is freely generated, as a Σ_n-module, by the following trees :



(3.18)

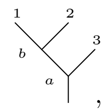
where $1 \leq l \leq k \leq n$.

Proof. Let $n \in \mathbb{N}^*$. Consider the following map :

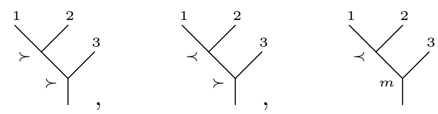
$$\Omega : \left\{ \begin{array}{rcl} \widetilde{\mathcal{BG}}^!(n) & \rightarrow & Vect(\text{forests } \in \mathbf{H}^!_{BG} \text{ of degree } n) \subseteq \mathbf{M}^!_{BG} \\ p & \rightarrow & p.(\bullet, \ldots, \bullet). \end{array} \right. \tag{3.19}$$

By proposition 55, Ω is surjective. So $dim(\mathcal{BG}^!(n)) \geq f_n^{\mathbf{H}^!_{BG}} n! = \frac{n(n+1)!}{2}$.

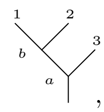
Moreover, from relations $r_1, \ldots, r_6$ and r_{10}, r_{11}, r_{12}, we obtain that $\mathcal{BG}^!(3)$ is generated by the trees of the following form :



with $a, b \in \{\prec, \succ, m\}$. Using relations r_7, r_8, r_9,

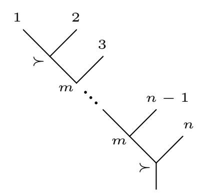


are eliminated. We deduce that $\mathcal{BG}^!(3)$ is generated by the trees of the following form :

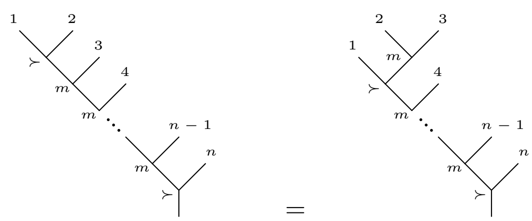


with $(a, b) \in \{(\succ, m), (m, \succ), (m, m), (\prec, \succ), (\prec, m), (\prec, \prec)\}$.

Let us prove that for all $n \geq 3$, the following tree is zero :



By induction on n. If $n = 3$ this tree is zero with the relation r_7. If $n \geq 4$,



with the relation r_2 and this tree is zero by induction hypothesis.

So $\mathcal{BG}^!(n)$ is generated, as a Σ_n-module, by the $\frac{n(n+1)}{2}$ trees given by the formula (3.18) and for all $n \in \mathbb{N}$, $dim(\mathcal{BG}^!(n)) \leq \dfrac{n(n+1)!}{2}$.

We deduce that $dim(\mathcal{BG}^!(n)) = \frac{n(n+1)!}{2}$. So Ω is bijective and $\mathbf{M}^!_{BG}$ is the free $\mathcal{BG}^!$-algebra generated by $\bullet$. Moreover $\mathcal{BG}^!(n)$ is freely generated, as a Σ_n-module, by the trees given to the formula (3.18), by equality of dimensions. □

We give some numerical values :

n	1	2	3	4	5	6	7	8	9	10
$dim\left(\widetilde{\mathcal{BG}}^!(n)\right)$	1	3	6	10	15	21	28	36	45	55
$dim\left(\mathcal{BG}^!(n)\right)$	1	6	36	240	1800	15120	141120	1451520	16329600	199584000

These are the sequences A000217 and A001286 in [Slo].

Remark. We denote by $F_{\mathcal{BG}}(x)$ and $F_{\mathcal{BG}^!}(x)$ the generating series associated to operads $\mathcal{BG}$ and $\mathcal{BG}^!$. Using proposition 54, $F_{\mathcal{BG}^!}(x) = \frac{x}{(1-x)^3}$. Moreover, by proposition 43 (see the proof), $F_{\mathcal{BG}}(x) = \frac{T(x)}{1-T(x)}$ where $T(x) = (x - 2x^2 + x^3)^{-1}$. So $F_{\mathcal{BG}}^{-1}(x) = \frac{x}{(1+x)^3}$ and we have :

$$F_{\mathcal{BG}}\left(-F_{\mathcal{BG}^!}(-x)\right) = x.$$

This result is also a consequence of theorem 61.

As the map Ω defined in formula (3.19) is bijective, we can identify $F \in \mathbb{F}^!_{BG}(n)$ and $\Omega^{-1}(F) \in \mathcal{BG}^!(n)$. We now describe the composition of $\mathcal{BG}^!$ in term of forests :

Theorem 57 *The composition $\circ$ of $\mathcal{BG}^!$ in the basis of forests belong to $\mathbb{F}^!_{BG} \setminus \{1\}$ can be inductively defined in this way :*

$$\begin{array}{rcl} \bullet \circ (H) &=& H, \\ FG \circ (H_1, \ldots, H_{|F|+|G|}) &=& \begin{cases} 0 \text{ if the forest } H_1 \ldots H_{|F|+|G|} \notin \mathbb{F}^!_{BG}, \\ F \circ (H_1, \ldots, H_{|F|}) G \circ (H_{|F|+1}, \ldots, H_{|F|+|G|}) \text{ if not,} \end{cases} \\ B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \circ (H_1, \ldots, H_{p+q+1}) &=& \begin{cases} 0 \text{ if there exists } i \neq p+1 \text{ such that } H_i \neq \bullet, \\ ((\underbrace{\bullet \ldots \bullet}_{p\times}) \succ H_{p+1}) \prec (\underbrace{\bullet \ldots \bullet}_{q\times}) \text{ if not.} \end{cases} \end{array}$$

Proof. This is the same proof as for theorem 50. $\square$

We denote by $\mathcal{BG}^!(V)$ the free $\mathcal{BG}^!$-algebra over a vector space V. Because the operad $\mathcal{BG}^!$ is regular, we get the following result :

Proposition 58 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{BG}^!$-algebra on V is*

$$\mathcal{BG}^!(V) = \bigoplus_{n \geq 1} \mathbb{K}\left(\mathbb{F}^!_{BG}(n)\right) \otimes V^{\otimes n},$$

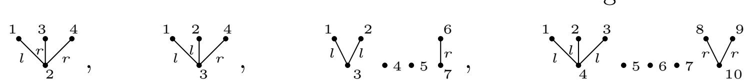
equipped with the following binary operations : for all $F \in \mathbb{F}^!_{BG}(n)$, $G \in \mathbb{F}^!_{BG}(m)$, $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \ldots \otimes w_m \in V^{\otimes m}$,

$$\begin{array}{rcl} (F \otimes v_1 \otimes \ldots \otimes v_n) * (G \otimes w_1 \otimes \ldots \otimes w_m) &=& (FG \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m), \\ (F \otimes v_1 \otimes \ldots \otimes v_n) \succ (G \otimes w_1 \otimes \ldots \otimes w_m) &=& (F \succ G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m), \\ (F \otimes v_1 \otimes \ldots \otimes v_n) \prec (G \otimes w_1 \otimes \ldots \otimes w_m) &=& (F \prec G \otimes v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m). \end{array}$$

Remark. Introduce an order relation on the set of the vertices of a forest $\in \mathbb{F}^!_{BG}$. Let $F = F_1 \ldots F_n \in \mathbb{F}^!_{BG} \setminus \{1\}$ and s, s' be two vertices of F. Then $s \leq s'$ if one of these assertions is satisfied :

1. $s \in V(F_i)$, $s' \in V(F_j)$ and $i < j$.
2. $s, s' \in V(F_i)$ with $F_i = B_{BG}(G_1 \ldots G_p \otimes H_1 \ldots H_q)$ (the G_k's and the H_k's are equal to $\bullet$) and :
   (a) $s \in V(G_k)$, $s' \in V(G_l)$ and $k < l$.
   (b) $s \in V(G_k)$ and s' is the root of F_i.
   (c) s is the root of F_i and $s' \in V(H_k)$.
   (d) $s \in V(H_k)$, $s' \in V(H_l)$ and $k < l$.

For instance, we give the order relation on the vertices for the following forests :



So we can see an element $(F \otimes v_1 \otimes \ldots \otimes v_n) \in \mathcal{BG}^!(V)$ as the forest F where the vertex i, for the previous order relation, is decorated by v_i.

3.2.2 Homology of $\mathcal{BG}^!$-algebras

Definition 59 *By taking the elements of V homogenous of degree 1, $\mathcal{BG}^!(V)$ is naturaly graduated. We define on $\mathcal{BG}^!(V)$ three coproducts given in the following way : if $F = F_1 \ldots F_k \in \mathbb{F}^!_{BG}(n)$ and $v = v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$, with*

– *if $k = 1$,*

$$F = B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes \underbrace{\bullet \ldots \bullet}_{q\times}),$$

– *if* $k \geq 2$,

$$F_1 = B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1), F_k = B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \text{ and for all } 2 \leq i \leq k-1, F_i = \bullet,$$

$(p+q+k=n)$ then

$$\begin{array}{rcl}
\Delta(F \otimes v) & = & \displaystyle\sum_{i=0}^{k} (F_1 \ldots F_i \otimes v_1 \otimes \ldots \otimes v_{p+i}) \otimes (F_{i+1} \ldots F_k \otimes v_{p+i+1} \otimes \ldots \otimes v_n),\\
\Delta_{\succ}(F \otimes v) & = & \displaystyle\sum_{i=0}^{p} (\underbrace{\bullet \ldots \bullet}_{i\times} \otimes v_1 \otimes \ldots \otimes v_i) \otimes (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p-i\times} \otimes 1) F_2 \ldots F_k \otimes v_{i+1} \otimes \ldots \otimes v_n),\\
\Delta_{\prec}(F \otimes v) & = & \displaystyle\sum_{i=0}^{q} (F_1 \ldots F_{k-1} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q-i\times}) \otimes v_1 \otimes \ldots \otimes v_{n-i}) \otimes (\underbrace{\bullet \ldots \bullet}_{i\times} \otimes v_{n-i+1} \otimes \ldots \otimes v_n).
\end{array}$$

Definition 60 *Let $(A, *, \succ, \prec)$ be a $\mathcal{BG}$-algebra. We define a differential $d : \mathcal{BG}^!(A)(n) \to \mathcal{BG}^!(A)(n-1)$ uniquely determined by the following conditions :*

1. *For all $a \in A$, $d(\bullet \otimes a) = 0$.*
2. *For all $a, b \in A$, $d(\bullet\bullet \otimes a \otimes b) = a * b$, $d(\overset{\bullet}{\bullet}{}^{l} \otimes a \otimes b) = a \succ b$ and $d(\overset{\bullet}{\bullet}{}^{r} \otimes a \otimes b) = a \prec b$.*
3. *Let $\theta : \mathcal{BG}^!(A) \to \mathcal{BG}^!(A)$ be the following map :*

$$\theta : \left\{ \begin{array}{rcl} \mathcal{BG}^!(A) & \to & \mathcal{BG}^!(A) \\ x & \to & (-1)^{degree(x)} x \text{ for all homogeneous } x. \end{array} \right.$$

Then d is a θ-coderivation : for all $x \in \mathcal{BG}^!(A)$,

$$\begin{array}{rcl}
\Delta(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta(x),\\
\Delta_{\succ}(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta_{\succ}(x),\\
\Delta_{\prec}(d(x)) & = & (d \otimes id + \theta \otimes d) \circ \Delta_{\prec}(x).
\end{array}$$

So, d is the map which sends the element $(B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-2\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \otimes v_1 \otimes \ldots \otimes v_n)$, where $p, q, k \in \mathbb{N}$, $k \geq 1$ and $p+q+k=n$ (if $k=1$, the element is $(B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \otimes v_1 \otimes \ldots \otimes v_n)$), to

$$\begin{array}{rl}
& \displaystyle\sum_{i=1}^{p-1} (-1)^{i-1} (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p-1\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-2\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \otimes v_1 \otimes \ldots \otimes v_i * v_{i+1} \otimes \ldots \otimes v_n)\\
+ & (-1)^{p-1} (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p-1\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-2\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \otimes v_1 \otimes \ldots \otimes v_p \succ v_{p+1} \otimes \ldots \otimes v_n)\\
+ & \displaystyle\sum_{i=p+1}^{p+k-1} (-1)^{i-1} (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-3\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q\times}) \otimes v_1 \otimes \ldots \otimes v_i * v_{i+1} \otimes \ldots \otimes v_n)\\
+ & (-1)^{p+k-1} (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-2\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q-1\times}) \otimes v_1 \otimes \ldots \otimes v_{p+k} \prec v_{p+k+1} \otimes \ldots \otimes v_n)\\
+ & \displaystyle\sum_{i=p+k+1}^{n-1} (-1)^{i-1} (B_{BG}(\underbrace{\bullet \ldots \bullet}_{p\times} \otimes 1) \underbrace{\bullet \ldots \bullet}_{k-2\times} B_{BG}(1 \otimes \underbrace{\bullet \ldots \bullet}_{q-1\times}) \otimes v_1 \otimes \ldots \otimes v_i * v_{i+1} \otimes \ldots \otimes v_n).
\end{array}$$

The homology of this complex will be denoted by $H_*(A)$. More clearly, for all $n \in \mathbb{N}$:

$$H_n(A) = \frac{Ker\left(d_{|\mathcal{BG}^!(A)(n+1)}\right)}{Im\left(d_{|\mathcal{BG}^!(A)(n+2)}\right)}$$

Examples. Let $a, b, c \in A$. Then $d(\bullet \otimes a) = 0$, $d(\bullet\bullet \otimes a \otimes b) = a * b$, $d(\overset{\bullet}{\bullet}_{l} \otimes a \otimes b) = a \succ b$ and $d(\overset{\bullet}{\bullet}_{r} \otimes a \otimes b) = a \prec b$. In degree 3,

$$
\begin{array}{rcl}
d(\bullet\bullet\bullet \otimes a \otimes b \otimes c) & = & -(\bullet\bullet \otimes a \otimes (b * c)) + (\bullet\bullet \otimes (a * b) \otimes c) \\
d(\overset{\bullet}{\bullet}_{l}\ \bullet \otimes a \otimes b \otimes c) & = & (\bullet\bullet \otimes (a \succ b) \otimes c) - (\overset{\bullet}{\bullet}_{l} \otimes a \otimes (b * c)) \\
d(\bullet\, \overset{\bullet}{\bullet}_{r} \otimes a \otimes b \otimes c) & = & (\overset{\bullet}{\bullet}_{r} \otimes (a * b) \otimes c) - (\bullet\bullet \otimes a \otimes (b \prec c)) \\
d({}^{l}\mathsf{V}_{l} \otimes a \otimes b \otimes c) & = & -(\overset{\bullet}{\bullet}_{l} \otimes a \otimes (b \succ c)) + (\overset{\bullet}{\bullet}_{l} \otimes (a * b) \otimes c) \\
d({}^{l}\mathsf{V}_{r} \otimes a \otimes b \otimes c) & = & -(\overset{\bullet}{\bullet}_{l} \otimes a \otimes (b \prec c)) + (\overset{\bullet}{\bullet}_{r} \otimes (a \succ b) \otimes c) \\
d({}^{r}\mathsf{V}_{r} \otimes a \otimes b \otimes c) & = & -(\overset{\bullet}{\bullet}_{r} \otimes a \otimes (b * c)) + (\overset{\bullet}{\bullet}_{r} \otimes (a \prec b) \otimes c)
\end{array}
$$

So, we obtain

$$
\begin{array}{rcl}
d^2(\bullet\bullet\bullet \otimes a \otimes b \otimes c) & = & -a * (b * c) + (a * b) * c \\
d^2(\overset{\bullet}{\bullet}_{l}\ \bullet \otimes a \otimes b \otimes c) & = & (a \succ b) * c - a \succ (b * c) \\
d^2(\bullet\, \overset{\bullet}{\bullet}_{r} \otimes a \otimes b \otimes c) & = & (a * b) \prec c - a * (b \prec c) \\
d^2({}^{l}\mathsf{V}_{l} \otimes a \otimes b \otimes c) & = & -a \succ (b \succ c) + (a * b) \succ c \\
d^2({}^{l}\mathsf{V}_{r} \otimes a \otimes b \otimes c) & = & -a \succ (b \prec c) + (a \succ c) \prec c \\
d^2({}^{r}\mathsf{V}_{r} \otimes a \otimes b \otimes c) & = & -a \prec (b * c) + (a \prec b) \prec c
\end{array}
$$

Hence, the nullity of d^2 is equivalent to the six relations defining a $\mathcal{BG}$-algebra (see formula (3.7)). In particular :

$$
H_0(A) = \frac{A}{A * A + A \succ A + A \prec A}
$$

3.2.3 The bigraft operad is Koszul

In this section, we use the rewriting method to prove that an operad is Koszul described in [LV12], see also [DK10, Hof10]. This is a short algorithmic method, based on the rewriting rules given by the relations.

Theorem 61 *The operad $\mathcal{BG}$ is Koszul.*

Proof. We consider $\widetilde{\mathcal{BG}}^{!} = \mathcal{P}(E, R)$ the nonsymmetric operad associated with $\mathcal{BG}^{!}$, where E is concentrated in degree 2 with $E(2) = \mathbb{K}m \oplus \mathbb{K} \succ \oplus \mathbb{K} \prec$ and R is concentrated in degree 3 with $R(3) \subseteq \mathcal{P}_E(3)$ the subspace generated by r_i, for all $i \in \{1, \ldots, 12\}$, defined in formula (3.16).

We use the lexicographical order and we set $\succ < m < \prec$. So, we can give the leading term for each r_i :

r_1	r_2	r_3	r_4	r_5	r_6
$\succ \circ (m, I)$	$m \circ (\succ, I)$	$\prec \circ (\prec, I)$	$\prec \circ (m, I)$	$\prec \circ (\succ, I)$	$m \circ (m, I)$

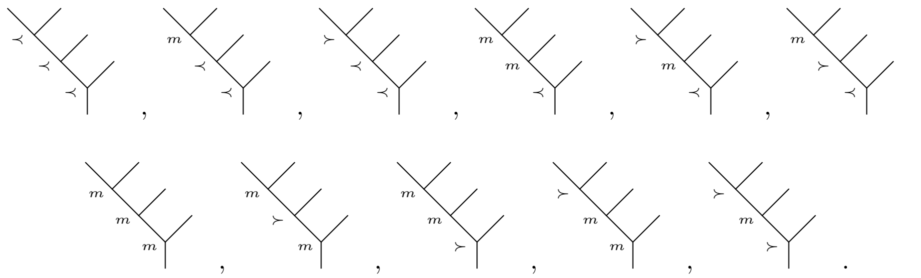
r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
$\succ \circ (\succ, I)$	$\succ \circ (\prec, I)$	$m \circ (\prec, I)$	$\prec \circ (I, \prec)$	$\prec \circ (I, \succ)$	$m \circ (I, \succ)$

Observe that each relation gives rise to a rewriting rule in the operad $\widetilde{\mathcal{BG}}^{!}$:

$$
\begin{array}{rcl|rcl}
\succ \circ (m, I) & \mapsto & \succ \circ (I, \succ) & \succ \circ (\succ, I) & \mapsto & 0 \\
m \circ (\succ, I) & \mapsto & \succ \circ (I, m) & \succ \circ (\prec, I) & \mapsto & 0 \\
\prec \circ (\prec, I) & \mapsto & \prec \circ (I, m) & m \circ (\prec, I) & \mapsto & 0 \\
\prec \circ (m, I) & \mapsto & m \circ (I, \prec) & \prec \circ (I, \prec) & \mapsto & 0 \\
\prec \circ (\succ, I) & \mapsto & \succ \circ (I, \prec) & \prec \circ (I, \succ) & \mapsto & 0 \\
m \circ (m, I) & \mapsto & m \circ (I, m) & m \circ (I, \succ) & \mapsto & 0
\end{array}
$$

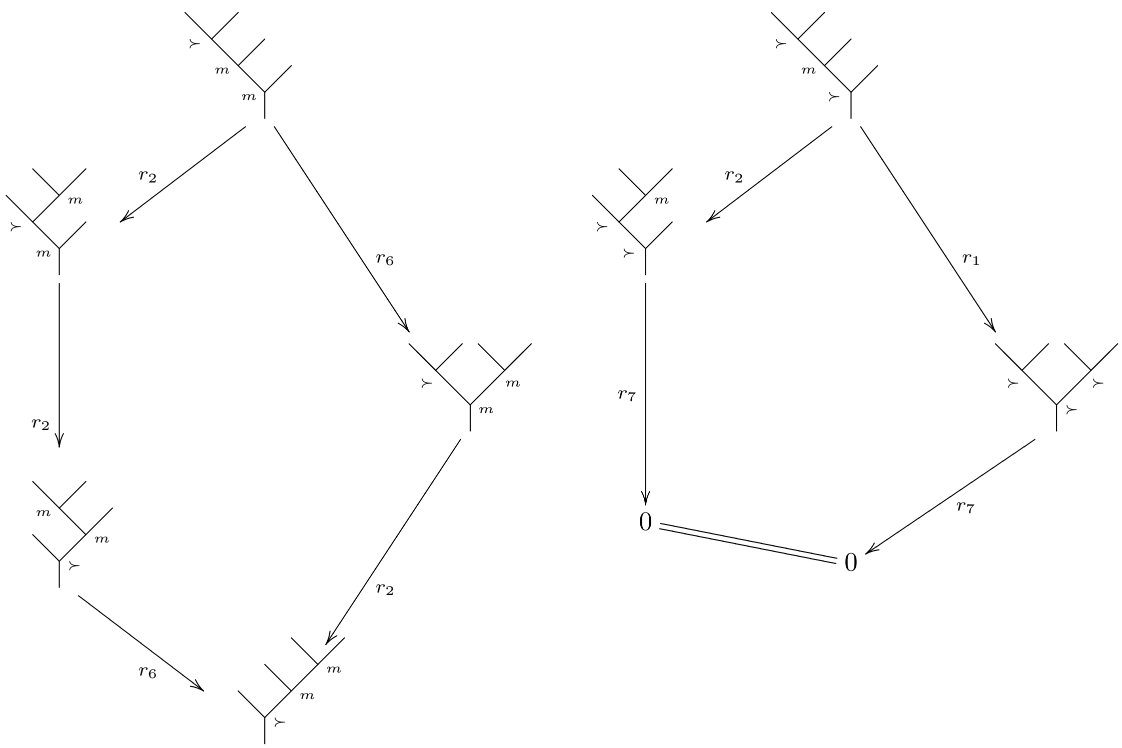
Given three operations $\in \{\succ, m, \prec\}$, one can compose them along 5 different ways : they correspond to the 5 planar binary trees with 4 leaves. Such a monomial, that is to say a decorated planar binary tree, is called critical if the two sub-trees with 3 leaves are leading terms.

We have 11 criticals monomials :



There are at least two ways of rewriting a critical monomial ad libitum, that is, until no rewriting rule is applicable any more. If all these ways lead to the same element, then the critical monomial is said to be confluent.

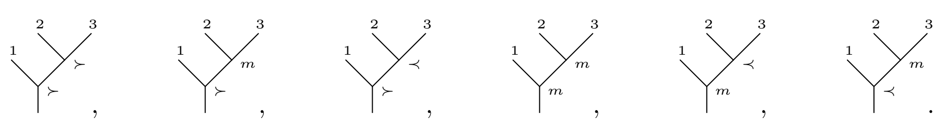
We can show that all critical monomials are confluent in the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$. We present for example the confluent graph for the last two criticals monomials :



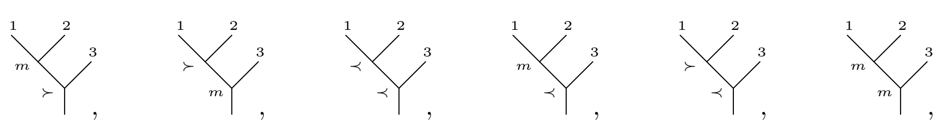
We can construct in the same way the confluent graph for each of the nine other criticals monomials.

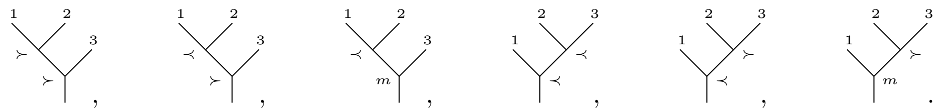
Then, with [LV12], we deduce that the nonsymmetric operad $\widetilde{\mathcal{BG}}^!$ is Koszul. So the operad $\mathcal{BG}$ is Koszul. □

We can give the quadratic part of a PBW basis of $\mathcal{BG}^!$:



The quadratic part of a PBW basis of its dual $\left(\mathcal{BG}^!\right)^! = \mathcal{BG}$ is :





Recall that $H_*(.)$ is the homology of the complex defined in the section 3.2.2. Then we deduce of the previous theorem the following result :

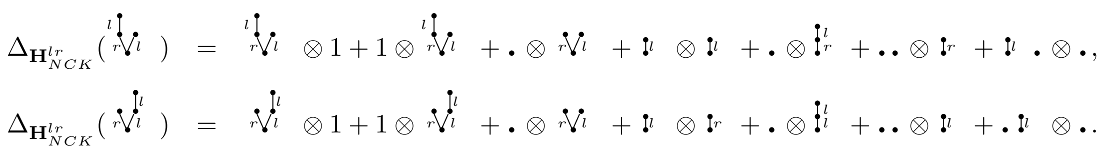
Corollary 62 *Let $N \geq 1$ and $(A, *, \succ, \prec)$ be the free $\mathcal{BG}$-algebra generated by N elements. Then $H_0(A)$ is N-dimensional and if $n \geq 1$, $H_n(A) = (0)$.*

3.3 Hopf algebra structure on the free bigraft algebra

In section 3.1.3, we have defined a decorated version $\mathbf{H}^{lr}_{NCK}$ of $\mathbf{H}_{NCK}$. We can define on $\mathbf{H}^{lr}_{NCK}$ a Hopf algebra structure. If $F \in \mathbf{H}^{lr}_{NCK}$ and $\boldsymbol{v} \models V(F)$, then $Lea_{\boldsymbol{v}}(F)$ and $Roo_{\boldsymbol{v}}(F)$ are naturally planar rooted forests with their edges decorated by l and r. So $\mathbf{H}^{lr}_{NCK}$ is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any forest $F \in \mathbf{H}^{lr}_{NCK}$ by :

$$\Delta_{\mathbf{H}^{lr}_{NCK}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \|\models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example :



Note that, unlike the cases of $(\mathbf{H}_{CK}, B_{CK})$ and $(\mathbf{H}_{NCK}, B_{NCK})$ (see theorems 1 and 2), this Hopf algebra structure on $\mathbf{H}^{lr}_{NCK}$ is not "universal" (with respect to the B_{BG}-operator). We will show that :

- We can define a "universal" Hopf algebra structure on the free bigraft algebra on one generator $\mathbf{H}_{BG}$ equipped with the operator B_{BG}.
- With this Hopf algebra structure, $\mathbf{H}_{BG}$ is a Hopf subalgebra of $\mathbf{H}^{lr}_{NCK}$.

3.3.1 Universal Hopf algebra on $\mathbf{H}_{BG}$

Let us prove that $(\mathbf{H}_{BG}, B_{BG})$ is an initial object in the category of couples (A, L) where A is an algebra and $L : A \otimes A \longrightarrow A$ any $\mathbb{K}$-linear map :

Lemma 63 *Let A be any algebra and let $L : A \otimes A \to A$ be a $\mathbb{K}$-linear map. Then there exists a unique algebra morphism $\phi : \mathbf{H}_{BG} \to A$ such that $\phi \circ B_{BG} = L \circ (\phi \otimes \phi)$.*

Proof. *Existence.* We define an element $a_F \in A$ for any forest $F \in \mathbf{H}_{BG}$ by induction on the degree of F as follows :

1. $a_1 = 1_A$.
2. If F is a tree, there exists two forests $G, H \in \mathbf{H}_{BG}$ with $|G| + |H| = |F| - 1$ such that $F = B_{BG}(G \otimes H)$; then $a_F = L(a_G \otimes a_H)$.
3. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}_{BG}$ and $|F_i| < |F|$; then $a_F = a_{F_1} \dots a_{F_n}$.

Let $\phi : \mathbf{H}_{BG} \to A$ be the unique linear morphism such that $\phi(F) = a_F$ for any forest F. Given two forests F and G, let us prove that $\phi(FG) = \phi(F)\phi(G)$. If $F = 1$ or $G = 1$, this is trivial. If not, $F = F_1 \dots F_n$ and $G = G_1 \dots G_m \neq 1$ and

$$\phi(FG) = a_{F_1} \dots a_{F_n} a_{G_1} \dots a_{G_m} = \phi(F)\phi(G).$$

Therefore ϕ is an algebra morphism. On the other hand, for all forests F, G,

$$\phi \circ B_{BG}(F \otimes G) = a_{B_{BG}(F \otimes G)} = L(a_F \otimes a_G) = L \circ (\phi \otimes \phi)(F \otimes G),$$

therefore $\phi \circ B_{BG} = L \circ (\phi \otimes \phi)$.

Uniqueness. Let $\psi : \mathbf{H}_{BG} \to A$ be an algebra morphism such that $\psi \circ B_{BG} = L \circ (\psi \otimes \psi)$. Let us prove that $\psi(F) = a_F$ for any forest F by induction on the degree of F. If $F = 1$, $\psi(F) = 1_A = a_1$. If $|F| \geq 1$, we have two cases :

1. If F is a tree then $F = B_{BG}(G \otimes H)$ with $G, H \in \mathbb{F}_{BG}$ and by induction hypothesis,
$$\psi(F) = \psi \circ B_{BG}(G \otimes H) = L \circ (\psi \otimes \psi)(G \otimes H) = L(a_G \otimes a_H) = a_F.$$
2. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}_{BG}$ and by induction hypothesis,
$$\psi(F) = \psi(F_1) \dots \psi(F_n) = a_{F_1} \dots a_{F_n} = a_F.$$

So $\phi(F) = \psi(F)$ for any forest F and $\phi = \psi$. □

We now consider
$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{H}_{BG} & \rightarrow & \mathbb{K} \\ F \in \mathbb{F}_{BG} & \mapsto & \delta_{F,1}. \end{array} \right.$$
ε is an algebra morphism satisfying $\varepsilon \circ B_{BG} = 0$.

Theorem 64 *Let $\Delta_{\mathbf{H}_{BG}} : \mathbf{H}_{BG} \rightarrow \mathbf{H}_{BG} \otimes \mathbf{H}_{BG}$ be the unique algebra morphism such that $\Delta_{\mathbf{H}_{BG}} \circ B_{BG} = L \circ (\Delta_{\mathbf{H}_{BG}} \otimes \Delta_{\mathbf{H}_{BG}})$ with*
$$L : \left\{ \begin{array}{rcl} (\mathbf{H}_{BG} \otimes \mathbf{H}_{BG}) \otimes (\mathbf{H}_{BG} \otimes \mathbf{H}_{BG}) & \rightarrow & \mathbf{H}_{BG} \otimes \mathbf{H}_{BG} \\ (x \otimes y) \otimes (z \otimes t) & \mapsto & B_{BG}(x \otimes z) \otimes \varepsilon(yt)1 + xz \otimes B_{BG}(y \otimes t). \end{array} \right.$$
Then $\Delta_{\mathbf{H}_{BG}}$ is a coassociative coproduct and $(\mathbf{H}_{BG}, m)$, endowed with this coproduct and the previous counit ε, is a graded connected Hopf algebra.

Proof. Thanks to lemma 63, $\Delta_{\mathbf{H}_{BG}}$ is well defined. To prove that $\mathbf{H}_{BG}$ is a graded connected Hopf algebra, we must prove that $\Delta_{\mathbf{H}_{BG}}$ is coassociative, counitary, and homogeneous of degree 0.

We first show that ε is a counit for $\Delta_{\mathbf{H}_{BG}}$. Let us prove that $(\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) = F$ for any forest $F \in \mathbb{F}_{BG}$ by induction on the degree of F. If $F = 1$,
$$(\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(1) = \varepsilon(1)1 = 1.$$

We use the Sweedler's notation for $\mathbf{H}_{BG}$: for all $x \in \mathbf{H}_{BG}$, $\Delta_{\mathbf{H}_{BG}}(x) = \sum_x x^{(1)} \otimes x^{(2)}$. Then, if $G, H \in \mathbb{F}_{BG}$,
$$\begin{aligned} \Delta_{\mathbf{H}_{BG}}(B_{BG}(G \otimes H)) &= \sum_{G,H} B_{BG}(G^{(1)} \otimes H^{(1)}) \otimes \varepsilon(G^{(2)}H^{(2)})1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}) \\ &= B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}). \end{aligned}$$

Note that, for all $x, y \in \mathbf{H}_{BG}$,
$$\Delta_{\mathbf{H}_{BG}} \circ B_{BG}(x \otimes y) = B_{BG}(x \otimes y) \otimes 1 + (m \otimes B_{BG}) \circ \Delta_{\mathbf{H}_{BG} \otimes \mathbf{H}_{BG}}(x \otimes y),$$
with $\Delta_{\mathbf{H}_{BG} \otimes \mathbf{H}_{BG}}(x \otimes y) = (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}} \otimes \Delta_{\mathbf{H}_{BG}})(x \otimes y)$ and τ the flip.

If F is a tree, $F = B_{BG}(G \otimes H)$ with $G, H \in \mathbb{F}_{BG}$. Then,
$$\begin{aligned} (\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) &= (\varepsilon \otimes Id) \left(B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}) \right) \\ &= \varepsilon \circ B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} \varepsilon(G^{(1)}H^{(1)}) B_{BG}(G^{(2)} \otimes H^{(2)}) \\ &= 0 + \sum_{G,H} B_{BG}(\varepsilon(G^{(1)})G^{(2)} \otimes \varepsilon(H^{(1)})H^{(2)}) \\ &= B_{BG}(G \otimes H) \\ &= F. \end{aligned}$$

If $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}_{BG}$. As $(\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}$ is an algebra morphism and using the induction hypothesis,

$$\begin{array}{rcl}(\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) & = & (\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F_1) \dots (\varepsilon \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F_n) \\ & = & F_1 \dots F_n \\ & = & F.\end{array}$$

Let us show that $(Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(F) = F$ for any forest $F \in \mathbb{F}_{BG}$ by induction. If $F = 1$,

$$(Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(1) = 1\varepsilon(1) = 1.$$

If F is a tree, $F = B_{BG}(G \otimes H)$ with $G, H \in \mathbb{F}_{BG}$ and then

$$\begin{array}{rcl}(Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(F) & = & (Id \otimes \varepsilon)\left(B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)})\right) \\ & = & B_{BG}(G \otimes H) + \sum_{G,H} G^{(1)}H^{(1)}\varepsilon \circ B_{BG}(G^{(2)} \otimes H^{(2)}) \\ & = & B_{BG}(G \otimes H) \\ & = & F.\end{array}$$

If $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}_{BG}$. As $(Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}$ is an algebra morphism and using the induction hypothesis,

$$\begin{array}{rcl}(Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(F) & = & (Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(F_1) \dots (Id \otimes \varepsilon) \circ \Delta_{\mathbf{H}_{BG}}(F_n) \\ & = & F_1 \dots F_n \\ & = & F.\end{array}$$

Therefore ε is a counit for $\Delta_{\mathbf{H}_{BG}}$.

Let us prove that $\Delta_{\mathbf{H}_{BG}}$ is coassociative. More precisely, we show that $(\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) = (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}(F)$ for any forest $F \in \mathbb{F}_{BG}$ by induction on the degree of F. If $F = 1$ this is obvious. If F is a tree, $F = B_{BG}(G \otimes H)$ with $G, H \in \mathbb{F}_{BG}$. Then

$$\begin{array}{rl} & (\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) \\ = & (\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}} \circ B_{BG}(G \otimes H) \\ = & (\Delta_{\mathbf{H}_{BG}} \otimes Id)\left(B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)})\right) \\ = & B_{BG}(G \otimes H) \otimes 1 \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}) \otimes 1 \\ & + \sum_{G,H,G^{(1)},H^{(1)}} (G^{(1)})^{(1)}(H^{(1)})^{(1)} \otimes (G^{(1)})^{(2)}(H^{(1)})^{(2)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}),\end{array}$$

$$\begin{array}{rl} & (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}(F) \\ = & (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}} \circ B_{BG}(G \otimes H) \\ = & (Id \otimes \Delta_{\mathbf{H}_{BG}})\left(B_{BG}(G \otimes H) \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)})\right) \\ = & B_{BG}(G \otimes H) \otimes 1 \otimes 1 + \sum_{G,H} G^{(1)}H^{(1)} \otimes B_{BG}(G^{(2)} \otimes H^{(2)}) \otimes 1 \\ & + \sum_{G,H,G^{(2)},H^{(2)}} G^{(1)}H^{(1)} \otimes (G^{(2)})^{(1)}(H^{(2)})^{(1)} \otimes B_{BG}((G^{(2)})^{(2)} \otimes (H^{(2)})^{(2)}).\end{array}$$

We conclude with the coassociativity of $\Delta_{\mathbf{H}_{BG}}$ applied to G and H. If F is not a tree, $F = F_1 \dots F_n$ with $F_i \in \mathbb{T}_{BG}$. As $(\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}$ and $(Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}$ are algebra morphisms and using the induction hypothesis,

$$\begin{array}{rl}
& (\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F) \\
= & (\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F_1) \ldots (\Delta_{\mathbf{H}_{BG}} \otimes Id) \circ \Delta_{\mathbf{H}_{BG}}(F_n) \\
= & \displaystyle\sum_{F_1,\ldots,F_n} \sum_{F_1^{(1)},\ldots,F_n^{(1)}} (F_1^{(1)})^{(1)} \ldots (F_n^{(1)})^{(1)} \otimes (F_1^{(1)})^{(2)} \ldots (F_n^{(1)})^{(2)} \otimes F_1^{(2)} \ldots F_n^{(2)} \\
= & \displaystyle\sum_{F_1,\ldots,F_n} \sum_{F_1^{(2)},\ldots,F_n^{(2)}} F_1^{(1)} \ldots F_n^{(1)} \otimes (F_1^{(2)})^{(1)} \ldots (F_n^{(2)})^{(1)} \otimes (F_1^{(2)})^{(2)} \ldots (F_n^{(2)})^{(2)} \\
= & (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}(F_1) \ldots (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}(F_n) \\
= & (Id \otimes \Delta_{\mathbf{H}_{BG}}) \circ \Delta_{\mathbf{H}_{BG}}(F).
\end{array}$$

Therefore $\Delta_{\mathbf{H}_{BG}}$ is coassociative.

Let us show that $\Delta_{\mathbf{H}_{BG}}$ is homogeneous of degree 0. Easy induction, using the fact that L is homogeneous of degree 1. Note that it can also be proved using proposition 66. As it is graded and connected, it has an antipode denoted by $S_{\mathbf{H}_{BG}}$. This ends the proof. $\Box$

Proposition 65 *For all forests $F \in \mathbf{H}_{BG}$, $\Delta_{\mathbf{H}_{BG}}(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(F)$.*

Proof. By induction on the degree n of F. If $n = 0$, $F = 1$ and this is obvious. Suppose that $n \geq 1$. We have two cases :

1. If $F = B_{BG}(G \otimes H)$ is a tree, with $G, H \in \mathbb{F}_{BG}$ such that $|G|, |H| < n$. Then

$$\begin{array}{rl}
& \Delta_{\mathbf{H}_{BG}}(F^\dagger) \\
= & \Delta_{\mathbf{H}_{BG}}(B_{BG}(H^\dagger \otimes G^\dagger)) \\
= & B_{BG}(H^\dagger \otimes G^\dagger) \otimes 1 + (m \otimes B_{BG}) \circ \Delta_{\mathbf{H}_{BG}}(H^\dagger \otimes G^\dagger) \\
= & B_{BG}(G \otimes H)^\dagger \otimes 1^\dagger \\
& +(m \otimes B_{BG}) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}) \otimes ((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}))(H \otimes G) \\
= & (\dagger \otimes \dagger)(B_{BG}(G \otimes H) \otimes 1) \\
& +((m \circ (\dagger \otimes \dagger)) \otimes (B_{BG} \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}} \otimes \Delta_{\mathbf{H}_{BG}})(H \otimes G) \\
= & (\dagger \otimes \dagger)(B_{BG}(G \otimes H) \otimes 1) \\
& +(\dagger \otimes \dagger) \circ (m \otimes B_{BG}) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}} \otimes \Delta_{\mathbf{H}_{BG}})(H \otimes G) \\
= & (\dagger \otimes \dagger)\,(B_{BG}(G \otimes H) \otimes 1 + (m \otimes B_{BG}) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}} \otimes \Delta_{\mathbf{H}_{BG}})(G \otimes H)) \\
= & (\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(F),
\end{array}$$

   using the induction hypothesis in the third equality.

2. If $F = GH$ is a forest, with $G, H \in \mathbb{F}_{BG} \setminus \{1\}$ such that $|G|, |H| < n$. Then

$$\begin{array}{rcl}
\Delta_{\mathbf{H}_{BG}}(F^\dagger) & = & \Delta_{\mathbf{H}_{BG}}(H^\dagger G^\dagger) \\
& = & (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}}(H^\dagger) \otimes \Delta_{\mathbf{H}_{BG}}(G^\dagger)) \\
& = & (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(H)) \otimes ((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(G))) \\
& = & ((m \circ (\dagger \otimes \dagger)) \otimes (m \circ (\dagger \otimes \dagger))) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}}(H) \otimes \Delta_{\mathbf{H}_{BG}}(G)) \\
& = & (\dagger \otimes \dagger) \circ (m \otimes m) \circ (\tau \otimes \tau) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}}(H) \otimes \Delta_{\mathbf{H}_{BG}}(G)) \\
& = & (\dagger \otimes \dagger) \circ (m \otimes m) \circ (Id \otimes \tau \otimes Id) \circ (\Delta_{\mathbf{H}_{BG}}(G) \otimes \Delta_{\mathbf{H}_{BG}}(H)) \\
& = & (\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(F),
\end{array}$$

   using the induction hypothesis in the third equality.

In all cases, $\Delta_{\mathbf{H}_{BG}}(F^\dagger) = (\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(F)$. $\Box$

We now give a combinatorial description of this coproduct :

Proposition 66 *Let $F \in \mathbb{F}_{BG}$. Then*

$$\Delta_{\mathbf{H}_{BG}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \Vdash V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

In other words, $\Delta_{\mathbf{H}_{BG}}$ is the restriction of $\Delta_{\mathbf{H}^{lr}_{NCK}} : \mathbf{H}^{lr}_{NCK} \rightarrow \mathbf{H}^{lr}_{NCK} \otimes \mathbf{H}^{lr}_{NCK}$ to $\mathbf{H}_{BG}$ and $\mathbf{H}_{BG}$ is a Hopf subalgebra of $\mathbf{H}^{lr}_{NCK}$.

Proof. Let $\Delta'_{\mathbf{H}_{BG}}$ be the unique algebra morphism from $\mathbf{H}_{BG}$ to $\mathbf{H}_{BG}\otimes\mathbf{H}_{BG}$ defined by the formula of proposition 66. It is easy to show that, if $F,G\in\mathbb{F}_{BG}$:

$$\left\{\begin{array}{rcl}\Delta'_{\mathbf{H}_{BG}}(1)&=&1\otimes 1,\\ \Delta'_{\mathbf{H}_{BG}}(B_{BG}(F\otimes G))&=&B_{BG}(F\otimes G)\otimes 1+(m\otimes B_{BG})\circ\Delta'_{\mathbf{H}_{BG}}(F\otimes G).\end{array}\right.$$

By unicity in theorem 64, $\Delta'_{\mathbf{H}_{BG}}=\Delta_{\mathbf{H}_{BG}}$. □

3.3.2 A Hopf pairing on $\mathbf{H}_{BG}$

Lemma 67 *Let (A,Δ_A) be a Hopf algebra and $L:A\otimes A\to A$ a linear morphism such that for all $a,b\in A$,*

$$\Delta_A\circ L(a\otimes b)=L(a\otimes b)\otimes 1+(m\otimes L)\circ\Delta_{A\otimes A}(a\otimes b).$$

Then the unique algebra morphism $\phi:\mathbf{H}_{BG}\to A$ such that $\phi\circ B_{BG}=L\circ(\phi\otimes\phi)$ (theorem 63) is a Hopf algebra morphism.

Proof. Let us prove that $\Delta_A\circ\phi(F)=(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}(F)$ for any forest $F\in\mathbf{H}_{BG}$ by induction on the degree of F. If $F=1$, this is obvious. If F is a tree, $F=B_{BG}(G\otimes H)$ with $G,H\in\mathbb{F}_{BG}$. Then

$$\begin{array}{rcl}\Delta_A\circ\phi(F)&=&\Delta_A\circ\phi\circ B_{BG}(G\otimes H)\\ &=&\Delta_A\circ L\circ(\phi\otimes\phi)(G\otimes H)\\ &=&L(\phi(G)\otimes\phi(H))\otimes 1+\displaystyle\sum_{\phi(G),\phi(H)}(\phi(G))^{(1)}(\phi(H))^{(1)}\otimes L((\phi(G))^{(2)}\otimes(\phi(H))^{(2)})\\ &=&\phi\circ B_{BG}(G\otimes H)\otimes\phi(1)+\displaystyle\sum_{G,H}\phi(G^{(1)}H^{(1)})\otimes\phi\circ B_{BG}(G^{(2)}\otimes H^{(2)})\\ &=&(\phi\otimes\phi)\left(B_{BG}(G\otimes H)\otimes 1+\displaystyle\sum_{G,H}G^{(1)}H^{(1)}\otimes B_{BG}(G^{(2)}\otimes H^{(2)})\right)\\ &=&(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}(F),\end{array}$$

using the induction hypothesis for the fourth equality. If F is not a tree, $F=F_1\dots F_n$ and then

$$\begin{array}{rcl}\Delta_A\circ\phi(F)&=&\Delta_A\circ\phi(F_1)\dots\Delta_A\circ\phi(F_n)\\ &=&(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}(F_1)\dots(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}(F_n)\\ &=&(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}(F),\end{array}$$

using the induction hypothesis for the second equality and $\Delta_A\circ\phi$ and $(\phi\otimes\phi)\circ\Delta_{\mathbf{H}_{BG}}$ are algebra morphisms for the first and third equality.

Let us prove that $\varepsilon\circ\phi=\varepsilon$. As $\varepsilon\circ\phi$ and ε are algebra morphisms, it suffices to show that $\varepsilon\circ\phi(F)=\varepsilon(F)$ for any tree $F\in\mathbb{T}_{BG}$. If $F=1$, this is obvious. If $F=B_{BG}(G\otimes H)$ with $G,H\in\mathbb{F}_{BG}$, $\varepsilon(F)=0$ and

$$\begin{array}{rcl}\varepsilon\circ\phi(F)&=&\varepsilon\circ\phi\circ B_{BG}(G\otimes H)\\ &=&\varepsilon\circ L\circ(\phi\otimes\phi)(G\otimes H).\end{array}$$

Let us prove that $\varepsilon\circ L=0$: for all $a,b\in A$,

$$\begin{array}{rcl}\Delta_A\circ L(a\otimes b)&=&L(a\otimes b)\otimes 1+\displaystyle\sum_{a,b}a^{(1)}b^{(1)}\otimes L(a^{(2)}\otimes b^{(2)})\\ (\varepsilon\otimes Id)\circ\Delta_A\circ L(a\otimes b)&=&\varepsilon\circ L(a\otimes b)1+\displaystyle\sum_{a,b}\varepsilon(a^{(1)}b^{(1)})L(a^{(2)}\otimes b^{(2)})\\ &=&\varepsilon\circ L(a\otimes b)+L(a\otimes b)\\ &=&L(a\otimes b)\end{array}$$

Therefore $\varepsilon\circ L=0$ and $\varepsilon\circ\phi=\varepsilon$. □

Let $\gamma:\mathbb{K}(\mathbb{T}_{BG})\to\mathbb{K}(\mathbb{F}_{BG})\otimes\mathbb{K}(\mathbb{F}_{BG})$ be the $\mathbb{K}$-linear map homogeneous of degree -1 such that for all $F,G\in\mathbb{F}_{BG}$, $\gamma(B_{BG}(F\otimes G))=(-1)^{|G|}F\otimes G$. From γ, we define the $\mathbb{K}$-linear map Γ by :

$$\Gamma:\left\{\begin{array}{rcl}\mathbf{H}_{BG}&\to&\mathbf{H}_{BG}\otimes\mathbf{H}_{BG},\\ T_1\dots T_n&\to&\Delta_{\mathbf{H}_{BG}}(T_1)\dots\Delta_{\mathbf{H}_{BG}}(T_{n-1})\gamma(T_n).\end{array}\right.$$

Lemma 68 *Let $\Gamma^* : \mathbf{H}_{BG}^{\circledast} \otimes \mathbf{H}_{BG}^{\circledast} \to \mathbf{H}_{BG}^{\circledast}$ be the transpose of Γ. Then the unique algebra morphism $\Phi : \mathbf{H}_{BG} \to \mathbf{H}_{BG}^{\circledast}$ such that $\Phi \circ B_{BG} = \Gamma^* \circ (\Phi \otimes \Phi)$ is a graded Hopf algebra morphism.*

Proof. Note that Γ is homogeneous of degree -1 and for all $x, y \in \mathbf{H}_{BG}$,

$$\Gamma(xy) = \Delta_{\mathbf{H}_{BG}}(x)\Gamma(y) + \varepsilon(y)\Gamma(x). \tag{3.20}$$

Then Γ^* is homogeneous of degree 1. Moreover, for all $f, g \in \mathbf{H}_{BG}^{\circledast}$,

$$\Delta_{\mathbf{H}_{BG}^{\circledast}}(\Gamma^*(f \otimes g)) = \Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta_{\mathbf{H}_{BG}^{\circledast} \otimes \mathbf{H}_{BG}^{\circledast}}(f \otimes g). \tag{3.21}$$

Indeed, if $x, y \in \mathbf{H}_{BG}$,

$$\begin{aligned}
\Delta_{\mathbf{H}_{BG}^{\circledast}}(\Gamma^*(f \otimes g))(x \otimes y) &= (f \otimes g)(\Gamma(xy)) \\
&= (f \otimes g)(\Delta_{\mathbf{H}_{BG}}(x)\Gamma(y) + \varepsilon(y)\Gamma(x)) \\
&= \varepsilon(y)(f \otimes g)(\Gamma(x)) + \Delta_{\mathbf{H}_{BG}^{\circledast} \otimes \mathbf{H}_{BG}^{\circledast}}(f \otimes g)(\Delta_{\mathbf{H}_{BG}}(x) \otimes \Gamma(y)) \\
&= \Big(\Gamma^*(f \otimes g) \otimes 1 + (m \otimes \Gamma^*) \circ \Delta_{\mathbf{H}_{BG}^{\circledast} \otimes \mathbf{H}_{BG}^{\circledast}}(f \otimes g)\Big)(x \otimes y).
\end{aligned}$$

With lemma 67 and formula (3.21), Φ is a Hopf algebra morphism.

Let us prove that it is homogeneous of degree 0. Let F be a forest in $\mathbf{H}_{BG}$ with n vertices. Then $\Phi(F)$ is homogeneous of degree n. If $n = 0$, then $F = 1$ and $\Phi(F) = 1$ is homogeneous of degree 0. Assume the result is true for all forests with $k < n$ vertices. If F is a tree, $F = B_{BG}(G \otimes H)$. Then $\Phi(F) = \Gamma^* \circ (\Phi(G) \otimes \Phi(H))$. By the induction hypothesis, $\Phi(G) \otimes \Phi(H)$ is homogeneous of degree $n-1$. As Γ^* is homogeneous of degree 1, $\Phi(F)$ is homogeneous of degree $n - 1 + 1 = n$. If $F = F_1 \ldots F_k$, $k \geq 2$. Then $\Phi(F) = \Phi(F_1) \ldots \Phi(F_k)$ is homogeneous of degree $|F_1| + \ldots + |F_k| = n$ by the induction hypothesis. □

Notations. For all $x \in \mathbf{H}_{BG}$, we note $\Delta_{\mathbf{H}_{BG}}(x) = \sum_x x^{(1)} \otimes x^{(2)}$ and $\Gamma(x) = \sum_x x_{(1)} \otimes x_{(2)}$.

Theorem 69 *For all $x, y \in \mathbf{H}_{BG}$, we set $\langle x, y \rangle = \Phi(x)(y)$. Then :*

1. *$\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathbf{H}_{BG}$.*
2. *$\langle xy, z \rangle = \sum_z \left\langle x, z^{(1)} \right\rangle \left\langle y, z^{(2)} \right\rangle$ for all $x, y, z \in \mathbf{H}_{BG}$.*
3. *$\langle B_{BG}(x \otimes y), z \rangle = \sum_z \langle x, z_{(1)} \rangle \langle y, z_{(2)} \rangle$ for all $x, y, z \in \mathbf{H}_{BG}$.*

Moreover :

4. *$\langle -, - \rangle$ is symmetric.*
5. *If x and y are homogeneous of different degrees, $\langle x, y \rangle = 0$.*
6. *$\langle S_{\mathbf{H}_{BG}}(x), y \rangle = \langle x, S_{\mathbf{H}_{BG}}(y) \rangle$ for all $x, y \in \mathbf{H}_{BG}$.*

Proof. First, as Φ is a Hopf algebra morphism, we have for all $x, y, z \in \mathbf{H}_{BG}$,

$$\langle 1, x \rangle = \Phi(1)(x) = \varepsilon(x) = \varepsilon \circ \Phi(x) = \Phi(x)(1) = \langle x, 1 \rangle\,,$$

$$\begin{aligned}
\sum_z \left\langle x, z^{(1)} \right\rangle \left\langle y, z^{(2)} \right\rangle &= \sum_z \Phi(x)(z^{(1)})\Phi(y)(z^{(2)}) \\
&= (\Phi(x)\Phi(y))(z) \\
&= \Phi(xy)(z) \\
&= \langle xy, z \rangle\,, \\
\sum_x \left\langle x^{(1)}, y \right\rangle \left\langle x^{(2)}, z \right\rangle &= \sum_x \Phi(x^{(1)})(y)\Phi(x^{(2)})(z) \\
&= \sum_x \Phi(x)^{(1)}(y)\Phi(x)^{(2)}(z) \\
&= \Phi(x)(yz) \\
&= \langle x, yz \rangle
\end{aligned}$$

and

$$\langle S_{\mathbf{H}_{BG}}(x), y\rangle = \Phi(S_{\mathbf{H}_{BG}}(x))(y) = S^*_{\mathbf{H}_{BG}}(\Phi(x))(y) = \Phi(x)(S_{\mathbf{H}_{BG}}(y)) = \langle x, S_{\mathbf{H}_{BG}}(y)\rangle .$$

We obtain points 1, 2 and 6.

As Φ is homogeneous of degree 0, this implies point 5.

Let $x, y, z \in \mathbf{H}_{BG}$. Then :

$$\begin{aligned} \langle B_{BG}(x \otimes y), z\rangle &= \Phi \circ B_{BG}(x \otimes y)(z) \\ &= \Gamma^* \circ (\Phi(x) \otimes \Phi(y))(z) \\ &= (\Phi(x) \otimes \Phi(y))(\Gamma(z)) \\ &= \sum_z \left\langle x, z_{(1)}\right\rangle \left\langle y, z_{(2)}\right\rangle . \end{aligned}$$

We obtain point 3.

It remains to prove point 4, that is to say that $\langle -, -\rangle$ is symmetric. First we show that for all $x, y, z \in \mathbf{H}_{BG}$,

$$\sum_x \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle = \langle x, B_{BG}(y \otimes z)\rangle .$$

We can suppose by bilinearity that x, y and z are three forests. By induction on the degree n of x. If $n = 0$, then $x = 1$ and

$$\begin{aligned} &\sum_x \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle = \langle 0, y\rangle \langle 0, z\rangle = 0, \\ &\langle x, B_{BG}(y \otimes z)\rangle = \varepsilon(B_{BG}(y \otimes z)) = 0. \end{aligned}$$

If $n = 1$, then $x = \bullet$ and $\Gamma(\bullet) = \gamma(\bullet) = 1 \otimes 1$. Then

$$\sum_x \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle = \langle 1, y\rangle \langle 1, z\rangle = \delta_{y,1}\delta_{z,1}$$

$$\begin{aligned} \langle x, B_{BG}(y \otimes z)\rangle &= \langle B_{BG}(1 \otimes 1), B_{BG}(y \otimes z)\rangle \\ &= \sum_{B_{BG}(y\otimes z)} \left\langle 1, B_{BG}(y \otimes z)_{(1)}\right\rangle \left\langle 1, B_{BG}(y \otimes z)_{(2)}\right\rangle \\ &= \delta_{B_{BG}(y\otimes z),\bullet} = \delta_{y,1}\delta_{z,1}. \end{aligned}$$

Suppose that the result is true for every forest x of degree $< n$. Two cases are possible :

1. If x is a tree, $x = B_{BG}(x' \otimes x'')$. First, $\Gamma(x) = (-1)^{|x''|} x' \otimes x''$ so

$$\begin{aligned} \sum_x \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle &= (-1)^{|x''|} \langle x', y\rangle \langle x'', z\rangle \\ &= \begin{cases} 0 \text{ if } deg(x'') \neq deg(z), \\ (-1)^d \langle x', y\rangle \langle x'', z\rangle \text{ if } d = deg(x'') = deg(z). \end{cases} \end{aligned}$$

   Moreover,

$$\begin{aligned} \langle x, B_{BG}(y \otimes z)\rangle &= \langle B_{BG}(x' \otimes x''), B_{BG}(y \otimes z)\rangle \\ &= \sum_{B_{BG}(y\otimes z)} \left\langle x', B_{BG}(y \otimes z)_{(1)}\right\rangle \left\langle x'', B_{BG}(y \otimes z)_{(2)}\right\rangle \\ &= (-1)^{|z|} \langle x', y\rangle \langle x'', z\rangle \\ &= \begin{cases} 0 \text{ if } deg(x'') \neq deg(z), \\ (-1)^d \langle x', y\rangle \langle x'', z\rangle \text{ if } d = deg(x'') = deg(z). \end{cases} \end{aligned}$$

2. If x is a forest with at least two trees. Then x can be written $x = x'x''$, with the induction hypothesis

avalaible for x' and x''. Then

$$\begin{aligned}
\langle x, B_{BG}(y \otimes z)\rangle &= \langle x'x'', B_{BG}(y \otimes z)\rangle \\
&= \sum_{B_{BG}(y\otimes z)} \left\langle x', B_{BG}(y \otimes z)^{(1)}\right\rangle \left\langle x'', B_{BG}(y \otimes z)^{(2)}\right\rangle \\
&= \langle x', B_{BG}(y \otimes z)\rangle \langle x'', 1\rangle + \sum_{y,z} \left\langle x', y^{(1)}z^{(1)}\right\rangle \left\langle x'', B_{BG}(y^{(2)} \otimes z^{(2)})\right\rangle \\
&= \langle x', B_{BG}(y \otimes z)\rangle \varepsilon(x'') \\
&\quad + \sum_{y,z} \left(\sum_{x'} \left\langle x'^{(1)}, y^{(1)}\right\rangle \left\langle x'^{(2)}, z^{(1)}\right\rangle \right) \left(\sum_{x''} \left\langle x''_{(1)}, y^{(2)}\right\rangle \left\langle x''_{(2)}, z^{(2)}\right\rangle \right) \\
&= \langle x', B_{BG}(y \otimes z)\rangle \varepsilon(x'') \\
&\quad + \sum_{x',x''} \left(\sum_{y} \left\langle x'^{(1)}, y^{(1)}\right\rangle \left\langle x''_{(1)}, y^{(2)}\right\rangle \right) \left(\sum_{z} \left\langle x'^{(2)}, z^{(1)}\right\rangle \left\langle x''_{(2)}, z^{(2)}\right\rangle \right) \\
&= \sum_{x'} \left\langle x'_{(1)}, y\right\rangle \left\langle x'_{(2)}, z\right\rangle \varepsilon(x'') + \sum_{x',x''} \left\langle x'^{(1)}x''_{(1)}, y\right\rangle \left\langle x'^{(2)}x''_{(2)}, z\right\rangle \\
&= \sum_{x} \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle ,
\end{aligned}$$

by formula (3.20) for the seventh equality.

So for all $x, y, z \in \mathbf{H}_{BG}$,

$$\sum_{x} \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle = \langle x, B_{BG}(y \otimes z)\rangle .$$

Let us prove that $\langle -, -\rangle$ is symmetric. By induction on the degree n of x. If $n = 0$, then $x = 1$ and

$$\langle x, y\rangle = \langle 1, y\rangle = \varepsilon(y) = \langle y, 1\rangle = \langle y, x\rangle .$$

If $deg(x) \geq 1$, two cases are possible :

1. If x is a tree, $x = B_{BG}(x' \otimes x'')$. Then :

$$\begin{aligned}
\langle x, y\rangle &= \langle B_{BG}(x' \otimes x''), y\rangle \\
&= \sum_{y} \left\langle x', y_{(1)}\right\rangle \left\langle x'', y_{(2)}\right\rangle \\
&= \sum_{y} \left\langle y_{(1)}, x'\right\rangle \left\langle y_{(2)}, x''\right\rangle \\
&= \langle y, B_{BG}(x' \otimes x'')\rangle \\
&= \langle y, x\rangle .
\end{aligned}$$

2. If x is a forest with at least two trees, x can be written $x = x'x''$ with $deg(x'), deg(x'') < deg(x)$. Then :

$$\begin{aligned}
\langle x, y\rangle &= \langle x'x'', y\rangle \\
&= \sum_{y} \left\langle x', y^{(1)}\right\rangle \left\langle x'', y^{(2)}\right\rangle \\
&= \sum_{y} \left\langle y^{(1)}, x'\right\rangle \left\langle y^{(2)}, x''\right\rangle \\
&= \langle y, x'x''\rangle \\
&= \langle y, x\rangle .
\end{aligned}$$

□

Note that the assertions 1-3 entirely determine $\langle F, G\rangle$ for $F, G \in \mathbb{F}_{BG}$ by induction on the degree. Therefore $\langle -, -\rangle$ is the unique pairing satisfying the assertions 1-3. So we can give the following definition :

Definition 70 *Let $\gamma : \mathbb{K}(\mathbb{T}_{BG}) \to \mathbb{K}(\mathbb{F}_{BG}) \otimes \mathbb{K}(\mathbb{F}_{BG})$ be the $\mathbb{K}$-linear map homogeneous of degree -1 such that for all $F, G \in \mathbb{F}_{BG}$, $\gamma(B_{BG}(F \otimes G)) = (-1)^{|G|} F \otimes G$ and Γ the $\mathbb{K}$-linear map defined by :*

$$\Gamma : \left\{ \begin{array}{rcl} \mathbf{H}_{BG} & \to & \mathbf{H}_{BG} \otimes \mathbf{H}_{BG}, \\ T_1 \dots T_n & \to & \Delta_{\mathbf{H}_{BG}}(T_1) \dots \Delta_{\mathbf{H}_{BG}}(T_{n-1}) \gamma(T_n). \end{array} \right.$$

We note $\Delta_{\mathbf{H}_{BG}}(x) = \sum_x x^{(1)} \otimes x^{(2)}$ and $\Gamma(x) = \sum_x x_{(1)} \otimes x_{(2)}$. Then we define by induction on the degree a Hopf algebra pairing $\langle -, - \rangle : \mathbf{H}_{BG} \times \mathbf{H}_{BG} \to \mathbb{K}$ with the following assertions :

1. *for all $x \in \mathbf{H}_{BG}$, $\langle 1, x \rangle = \varepsilon(x)$,*
2. *for all $x, y, z \in \mathbf{H}_{BG}$, $\langle xy, z \rangle = \sum_z \left\langle x, z^{(1)} \right\rangle \left\langle y, z^{(2)} \right\rangle$,*
3. *for all $x, y, z \in \mathbf{H}_{BG}$, $\langle B_{BG}(x \otimes y), z \rangle = \sum_z \left\langle x, z_{(1)} \right\rangle \left\langle y, z_{(2)} \right\rangle$.*

Examples. Values of the pairing $\langle -, - \rangle$ for forests of degree ≤ 3 :

	$\bullet$
$\bullet$	1

	$\bullet\bullet$	$\vert_l$	$\vert_r$
$\bullet\bullet$	2	1	1
$\vert_l$	1	1	0
$\vert_r$	1	0	-1

	$\bullet\bullet\bullet$	$\bullet\vert_l$	$\vert_l\bullet$	$\bullet\vert_r$	$\vert_r\bullet$	${}_l\vee_l$	${}_l\vee_r$	${}_r\vee_r$	$\vert^l_l$	$\vert^r_l$	$\vert^l_r$	$\vert^r_r$
$\bullet\bullet\bullet$	6	3	3	3	3	1	2	1	1	1	1	1
$\bullet\vert_l$	3	2	2	1	1	1	1	0	1	1	0	0
$\vert_l\bullet$	3	2	2	1	1	0	1	1	1	0	1	0
$\bullet\vert_r$	3	1	1	0	0	0	-1	-1	0	0	-1	-1
$\vert_r\bullet$	3	1	1	0	0	1	1	1	0	-1	0	-1
${}_l\vee_l$	1	1	0	0	1	1	0	0	1	1	0	0
${}_l\vee_r$	2	1	1	-1	1	0	-1	0	0	0	0	0
${}_r\vee_r$	1	0	1	-1	1	0	0	2	0	0	1	1
$\vert^l_l$	1	1	1	0	0	1	0	0	1	0	0	0
$\vert^r_l$	1	1	0	0	-1	1	0	0	0	-1	0	0
$\vert^l_r$	1	0	1	-1	0	0	0	1	0	0	1	0
$\vert^r_r$	1	0	0	-1	-1	0	0	1	0	0	0	-1

Question. It is not difficult to see that $\langle -, - \rangle$ is nondegenerate in degree ≤ 3. We conjecture that it is nondegenerate for all degrees.

3.3.3 Relationship between the coproduct and the bigraft products

Is the coproduct just defined in the section 3.3.1 compatible with the bigraft products on $\mathbf{M}_{BG}$?

To answer this question, let's first give some results in the case of the right graft algebra $\mathbf{M}_{NCK}$. As $\mathcal{RG}$-algebras are not unitary objects, we need to extend the usual tensor product in order to obtain a copy of A and B in the tensor product of two vector spaces A and B :

$$A \overline{\otimes} B = (A \otimes \mathbb{K}) \oplus (A \otimes B) \oplus (\mathbb{K} \otimes B).$$

Note that if A, B, C are three vector spaces, then we have $(A \overline{\otimes} B) \overline{\otimes} C = A \overline{\otimes} (B \overline{\otimes} C)$.

Let A be a $\mathcal{RG}$-algebra. We extend $\prec: A \otimes A \to A$ to a map $\prec: A \overline{\otimes} A \to A$ in the following way : for all $a \in A$, $a \prec 1 = a$ and $1 \prec a = 0$. Moreover, we extend the product of A to a map from $(A \oplus \mathbb{K}) \otimes (A \oplus \mathbb{K})$ to $A \oplus \mathbb{K}$ by putting $1 * a = a * 1 = a$ for all $a \in A$ and $1 * 1 = 1$. Note that $1 \prec 1$ is not defined.

Recall the definition of a dipterous algebra (see [LR06]). A (right) dipterous algebra is a $\mathbb{K}$-vector space A equipped with two binary operations denoted by $*$ and $\prec$ satisfying the following relations : for all $x, y, z \in A$,

$$\begin{aligned} (x * y) * z &= x * (y * z), && (3.22)\\ (x \prec y) \prec z &= x \prec (y * z). && (3.23)\end{aligned}$$

Dipterous algebras do not have unit for the product $*$. If A and B are two dipterous algebras, we say that a $\mathbb{K}$-linear map $f : A \to B$ is a dipterous morphism if $f(x * y) = f(x) * f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by *Dipt*-alg the category of dipterous algebras. A $\mathcal{RG}$-algebra is also a dipterous algebra. We get the following canonical functor : $\mathcal{RG}$-alg $\to$ *Dipt*-alg.

Lemma 71 *Let A and B be two $\mathcal{RG}$-algebras. Then $A\overline{\otimes}B$ is a dipterous algebra with products defined in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{aligned} (a \otimes b) * (a' \otimes b') &= (a * a') \otimes (b * b'),\\ (a \otimes b) \prec (a' \otimes b') &= (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B,\\ (a \otimes 1) \prec (a' \otimes 1) &= (a \prec a') \otimes 1.\end{aligned}$$

Proof. The associativity of $* : (A\overline{\otimes}B) \otimes (A\overline{\otimes}B) \to A\overline{\otimes}B$ (that is to say the relation (3.22)) is obvious. We prove (3.23) : for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{aligned} (a \otimes b) \prec ((a' \otimes b') * (a'' \otimes b'')) &= (a \otimes b) \prec ((a' * a'') \otimes (b' * b''))\\ &= (a * (a' * a'')) \otimes (b \prec (b' * b''))\\ &= ((a * a') * a'') \otimes ((b \prec b') \prec b'')\\ &= ((a * a') \otimes (b \prec b')) \prec (a'' \otimes b'')\\ &= ((a \otimes b) \prec (a' \otimes b')) \prec (a'' \otimes b'').\end{aligned}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b'' = 1$ and $b' \in B$,

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes b') * (a'' \otimes 1)) &= (a \otimes 1) \prec ((a' * a'') \otimes b')\\ &= (a * (a' * a'')) \otimes (1 \prec b')\\ &= 0,\\ ((a \otimes 1) \prec (a' \otimes b')) \prec (a'' \otimes 1) &= ((a * a') \otimes (1 \prec b')) \prec (a'' \otimes 1)\\ &= 0.\end{aligned}$$

If $b = b' = 1$ and $b'' \in B$, then a and a' are not equal to 1 and

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes b'')) &= (a \otimes 1) \prec ((a' * a'') \otimes b'')\\ &= (a * (a' * a'')) \otimes (1 \prec b'')\\ &= 0,\\ ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes b'') &= ((a \prec a') \otimes 1) \prec (a'' \otimes b'')\\ &= ((a * a') \prec a'') \otimes (1 \prec b'')\\ &= 0.\end{aligned}$$

Finally if $b = b' = b'' = 1$, then a, a' and a'' are not equal to 1 and

$$\begin{aligned} (a \otimes 1) \prec ((a' \otimes 1) * (a'' \otimes 1)) &= (a \otimes 1) \prec ((a' * a'') \otimes 1)\\ &= (a \prec (a' * a'')) \otimes 1\\ &= ((a \prec a') \prec a'') \otimes 1\\ &= ((a \prec a') \otimes 1) \prec (a'' \otimes 1)\\ &= ((a \otimes 1) \prec (a' \otimes 1)) \prec (a'' \otimes 1).\end{aligned}$$

In all cases, the relation (3.23) is satisfied and $A\overline{\otimes}B$ is a dipterous algebra. $\square$

Remarks. Suppose that $A, B \neq \{0\}$. Then $(A\overline{\otimes}B, *, \prec)$ is a $\mathcal{RG}$-algebra if and only if $\prec: A \otimes A \to A$ and $\prec: B \otimes B \to B$ are zero. Indeed, if $A\overline{\otimes}B$ is a $\mathcal{RG}$-algebra then for all $a, a' \in A$ and $b, b' \in B$,

$$\begin{array}{rcl} (a \prec a') \otimes b & = & ((a \prec a') \otimes 1) * (1 \otimes b) \\ & = & ((a \otimes 1) \prec (a' \otimes 1)) * (1 \otimes b) \\ & = & (a \otimes 1) \prec (a' \otimes b) \\ & = & (a * a') \otimes (1 \prec b) \\ & = & 0, \end{array}$$

therefore, by taking $b \neq 0$, $a \prec a' = 0$ for all $a, a' \in A$. Moreover,

$$\begin{array}{rcl} a \otimes (b \prec b') & = & (a \otimes b) \prec (1 \otimes b') \\ & = & ((1 \otimes b) * (a \otimes 1)) \prec (1 \otimes b') \\ & = & (1 \otimes b) * (a \otimes (1 \prec b')) \\ & = & 0, \end{array}$$

therefore, by taking $a \neq 0$, $b \prec b' = 0$ for all $b, b' \in B$.

Reciprocally, if $\prec: A\otimes A \to A$ and $\prec: B\otimes B \to B$ are zero, it is clear that $(A\overline{\otimes}B, *, \prec)$ is a $\mathcal{RG}$-algebra.

Proposition 72 *For any tree $T \in \mathbf{M}_{NCK}$ and for any forest $F \in \mathbf{M}_{NCK}$,*

$$\Delta_{\mathbf{H}_{NCK}}(T \prec F) = \Delta_{\mathbf{H}_{NCK}}(T) \prec \Delta_{\mathbf{H}_{NCK}}(F).$$

In other words, $\Delta_{\mathbf{H}_{NCK}} : \mathbf{M}_{NCK} \to \mathbf{M}_{NCK}\overline{\otimes}\mathbf{M}_{NCK}$ is a dipterous morphism.

Proof. Let T and F are two planar trees $\in \mathbf{M}_{NCK}$. We note $\Delta_{\mathbf{H}_{NCK}}(T) = T\otimes 1+1\otimes T+\sum_T T^{(1)}\otimes T^{(2)}$ and $\Delta_{\mathbf{H}_{NCK}}(F) = F \otimes 1 + 1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}$. Then

$$\begin{array}{rl} & \Delta_{\mathbf{H}_{NCK}}(T) \prec \Delta_{\mathbf{H}_{NCK}}(F) \\ = & \left(T \otimes 1 + 1 \otimes T + \sum\limits_T T^{(1)} \otimes T^{(2)}\right) \prec \left(F \otimes 1 + 1 \otimes F + \sum\limits_F F^{(1)} \otimes F^{(2)}\right) \\ = & (T \prec F) \otimes 1 + T \otimes F + \sum\limits_T T^{(1)} F \otimes T^{(2)} + 1 \otimes (T \prec F) \\ & + \sum\limits_T T^{(1)} \otimes (T^{(2)} \prec F) + \sum\limits_F F^{(1)} \otimes (T \prec F^{(2)}) \\ & + \sum\limits_{T,F} T^{(1)} F^{(1)} \otimes (T^{(2)} \prec F^{(2)}) \\ = & \Delta_{\mathbf{H}_{NCK}}(T \prec F). \end{array}$$

If $F = F_1 \dots F_n \in \mathbf{M}_{NCK}$ is a forest and T is again a tree $\in \mathbf{M}_{NCK}$,

$$\begin{array}{rl} & \Delta_{\mathbf{H}_{NCK}}(T) \prec \Delta_{\mathbf{H}_{NCK}}(F) \\ = & \Delta_{\mathbf{H}_{NCK}}(T) \prec (\Delta_{\mathbf{H}_{NCK}}(F_1) \dots \Delta_{\mathbf{H}_{NCK}}(F_n)) \\ = & (\dots((\Delta_{\mathbf{H}_{NCK}}(T) \prec \Delta_{\mathbf{H}_{NCK}}(F_1)) \prec \Delta_{\mathbf{H}_{NCK}}(F_2)) \dots \prec \Delta_{\mathbf{H}_{NCK}}(F_{n-1})) \prec \Delta_{\mathbf{H}_{NCK}}(F_n) \\ = & \Delta_{\mathbf{H}_{NCK}}((\dots((T \prec F_1) \prec F_2) \dots \prec F_{n-1}) \prec F_n) \\ = & \Delta_{\mathbf{H}_{NCK}}(T \prec F), \end{array}$$

as $A\overline{\otimes}B$ is a dipterous algebra, for the second equality. $\square$

In fact, the right graft product is not fully compatible with the given coproduct. As $\mathbf{M}_{NCK}\overline{\otimes}\mathbf{M}_{NCK}$ is not a $\mathcal{RG}$-algebra, if $T \in \mathbf{M}_{NCK}$ is a forest, $\Delta_{\mathbf{H}_{NCK}}(T \prec F) \neq \Delta_{\mathbf{H}_{NCK}}(T) \prec \Delta_{\mathbf{H}_{NCK}}(F)$ in general. For example,

$$\begin{array}{rcl} \Delta_{\mathbf{H}_{NCK}}((\bullet\bullet) \prec \bullet) & = & \Delta_{\mathbf{H}_{NCK}}(\bullet\overset{\bullet}{\underset{\bullet}{|}}) \\ & = & \bullet\overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + 1 \otimes \bullet\overset{\bullet}{\underset{\bullet}{|}} + \overset{\bullet}{\underset{\bullet}{|}} \otimes \bullet + \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet\bullet \otimes \bullet + \bullet \otimes \bullet\bullet \\ \Delta_{\mathbf{H}_{NCK}}(\bullet\bullet) \prec \Delta_{\mathbf{H}_{NCK}}(\bullet) & = & (\bullet\bullet \otimes 1 + 1 \otimes \bullet\bullet + 2\bullet \otimes \bullet) \prec (\bullet \otimes 1 + 1 \otimes \bullet) \\ & = & \bullet\overset{\bullet}{\underset{\bullet}{|}} \otimes 1 + 1 \otimes \bullet\overset{\bullet}{\underset{\bullet}{|}} + \bullet \otimes \bullet\bullet + 2\bullet\bullet \otimes \bullet + 2\bullet \otimes \overset{\bullet}{\underset{\bullet}{|}}. \end{array}$$

We now focus on the case of bigraft algebras. We will encounter the same difficulties. As bigraft algebras are not objects with unit, we consider again the extended tensor product $A\overline{\otimes}B$ for A, B two $\mathcal{BG}$-algebras. If A is a bigraft algebra, we extend $\succ, \prec\colon A \otimes A \to A$ into maps $\succ, \prec\colon A\overline{\otimes}A \to A$ in the following way : for all $a \in A$,

$$a \succ 1 = 0, \qquad a \prec 1 = a, \qquad 1 \succ a = a, \qquad 1 \prec a = 0.$$

Moreover, we extend the product of A into a map from $(A \oplus \mathbb{K}) \otimes (A \oplus \mathbb{K})$ to $A \oplus \mathbb{K}$ by putting $1 * a = a * 1 = a$ for all $a \in A$ and $1 * 1 = 1$. Note that relations (3.7) are now satisfied on $A\overline{\otimes}A\overline{\otimes}A$.

We recall (see [Ler03]) that a pre-dendriform algebra is a $\mathbb{K}$-vector space A equipped with three binary operations denoted by $*$, $\succ$ and $\prec$ satisfying the four relations : for all $x, y, z \in A$,

$$\begin{array}{rcl}
(x * y) * z &=& x * (y * z),\\
(x * y) \succ y &=& x \succ (y \succ z),\\
(x \prec y) \prec z &=& x \prec (y * z),\\
(x \succ y) \prec z &=& x \succ (y \prec z).
\end{array}$$

In other words, $(A, *, \succ)$ and $(A, *, \prec)$ are dipterous algebras with the entanglement relation $(x \succ y) \prec z = x \succ (y \prec z)$.

Pre-dendriform algebras do not have unit for the product $*$. If A and B are two dipterous algebras, a $\mathbb{K}$-linear map $f : A \to B$ is a pre-dendriform morphism if $f(x*y) = f(x)*f(y)$, $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$. Let us denote by $PreDend$-alg the category of pre-dendriform algebras. A $\mathcal{BG}$-algebra is also a pre-dendriform algebra. We get the following canonical functor : $\mathcal{BG}$-alg $\to PreDend$-alg.

Lemma 73 *Let A and B be two bigraft algebras. Then $A\overline{\otimes}B$ is given a structure of pre-dendriform algebra in the following way : for all $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{array}{rcl}
(a \otimes b) * (a' \otimes b') &=& (a * a') \otimes (b * b'),\\
(a \otimes b) \succ (a' \otimes b') &=& (a * a') \otimes (b \succ b'), \ \textit{if } b \textit{ or } b' \in B,\\
(a \otimes 1) \succ (a' \otimes 1) &=& (a \succ a') \otimes 1,\\
(a \otimes b) \prec (a' \otimes b') &=& (a * a') \otimes (b \prec b'), \ \textit{if } b \textit{ or } b' \in B,\\
(a \otimes 1) \prec (a' \otimes 1) &=& (a \prec a') \otimes 1.
\end{array}$$

Proof. With lemma 71, $(A\overline{\otimes}B, *, \prec)$ is a right dipterous algebra. In the same way, $(A\overline{\otimes}B, *, \succ)$ is a left dipterous algebra. It remains to show the entanglement relation : for all $a, a', a'' \in A$ and $b, b', b'' \in B$,

$$\begin{array}{rcl}
((a \otimes b) \succ (a' \otimes b')) \prec (a'' \otimes b'') &=& ((a * a') * a'') \otimes ((b \succ b') \prec b'')\\
&=& (a * (a' * a'')) \otimes (b \succ (b' \prec b''))\\
&=& (a \otimes b) \succ ((a' \otimes b') \prec (a'' \otimes b'')).
\end{array}$$

This calculation is still true if b, b' or b'' is equal to 1 or if $b' = b'' = 1$ and $b \in B$. If $b = b' = 1$ and $b'' \in B$,

$$\begin{array}{rcl}
((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes b'') &=& ((a \succ a') * a'') \otimes (1 \prec b'')\\
&=& 0,\\
(a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes b'')) &=& (a \otimes 1) \succ ((a' * a'') \otimes (1 \prec b''))\\
&=& 0.
\end{array}$$

If $b' = b'' = 1$ and $b \in B$, it is the same calculation as previously. Finally if $b = b' = b'' = 1$,

$$\begin{array}{rcl}
((a \otimes 1) \succ (a' \otimes 1)) \prec (a'' \otimes 1) &=& ((a \succ a') \prec a'') \otimes 1\\
&=& (a \succ (a' \prec a'')) \otimes 1\\
&=& (a \otimes 1) \succ ((a' \otimes 1) \prec (a'' \otimes 1)).
\end{array}$$

$\square$

Theorem 74 *For all forests $F, G \in \mathbf{M}_{BG}$ and for all tree $T \in \mathbf{M}_{BG}$,*

$$\Delta_{\mathbf{H}_{BG}}(F \succ T \prec G) = \Delta_{\mathbf{H}_{BG}}(F) \succ \Delta_{\mathbf{H}_{BG}}(T) \prec \Delta_{\mathbf{H}_{BG}}(G). \tag{3.24}$$

In other words, $\Delta_{\mathbf{H}_{BG}} : \mathbf{M}_{BG} \to \mathbf{M}_{BG}\overline{\otimes}\mathbf{M}_{BG}$ is a pre-dendriform morphism.

Proof. With proposition 72, we have $\Delta_{\mathbf{H}_{BG}}(T \prec G) = \Delta_{\mathbf{H}_{BG}}(T) \prec \Delta_{\mathbf{H}_{BG}}(G)$. Moreover,

$$\begin{array}{rcl} \Delta_{\mathbf{H}_{BG}}(F \succ T) & = & \Delta_{\mathbf{H}_{BG}}((F^\dagger)^\dagger \succ (T^\dagger)^\dagger) \\ & = & (\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(T^\dagger \prec F^\dagger) \\ & = & (\dagger \otimes \dagger) \circ (\Delta_{\mathbf{H}_{BG}}(T^\dagger) \prec \Delta_{\mathbf{H}_{BG}}(F^\dagger)) \\ & = & (\dagger \otimes \dagger) \circ (((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(T)) \prec ((\dagger \otimes \dagger) \circ \Delta_{\mathbf{H}_{BG}}(F))) \\ & = & (\dagger \otimes \dagger) \circ (\dagger \otimes \dagger) \circ (\Delta_{\mathbf{H}_{BG}}(F) \succ \Delta_{\mathbf{H}_{BG}}(T)) \\ & = & \Delta_{\mathbf{H}_{BG}}(F) \succ \Delta_{\mathbf{H}_{BG}}(T), \end{array}$$

using propositions 65 and 48. As the entanglement relation is satisfied in $\mathbf{M}_{BG}$ and $\mathbf{M}_{BG} \overline{\otimes} \mathbf{M}_{BG}$, any parenthesizing of (3.24) gives the same relation. □

Remark. As $\mathbf{M}_{BG} \overline{\otimes} \mathbf{M}_{BG}$ is not a $\mathcal{BG}$-algebra, the relation (3.24) is not true if T is a forest in general.

3.4 A good triple of operads $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$

3.4.1 Good triple of operads and rigidity theorem for the right graft algebras

Recall some results on generalized bialgebra and good triple of operads (see [Lod08]).

Let $\mathcal{A}$ and $\mathcal{C}$ be two algebraic operads. A generalized bialgebra associated with $\mathcal{A}$ and $\mathcal{C}$, or $\mathcal{C}^c$-$\mathcal{A}$-bialgebra, is a $\mathbb{K}$-vector space H which is an $\mathcal{A}$-algebra, a $\mathcal{C}$-coalgebra, and such that the operations of $\mathcal{A}$ and the cooperations of $\mathcal{C}$ acting on H satisfy some compatibility relations.

Suppose that the following two hypothesis are fulfilled :
- There is a distributive compatibility relation for any pair (δ, μ) where μ is an operation and δ is a cooperation.
- The free $\mathcal{A}$-algebra $\mathcal{A}(V)$ over a $\mathbb{K}$-vector space V is equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure.

Then it determines an operad $\mathcal{P} := Prim_{\mathcal{C}}\mathcal{A}$ and a functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$. The operad $\mathcal{P}$ is the largest suboperad of $\mathcal{A}$ such that any $\mathcal{P}$-operation applied on primitive elements gives a primitive element. For any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H the inclusion $Prim(H) \hookrightarrow H$ becomes a morphism of $\mathcal{P}$-algebras. J.-L. Loday call this whole structure a *triple of operads* denoted by $(\mathcal{C}, \mathcal{A}, \mathcal{P})$.

The functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$ is a forgetful functor in the sense that the composition $\mathcal{A}\text{-alg} \xrightarrow{F} \mathcal{P}\text{-alg} \to \text{Vect}$ is the forgetful functor $\mathcal{A}\text{-alg} \to \text{Vect}$. This forgetful functor has a left adjoint denoted by $U : \mathcal{P}\text{-alg} \to \mathcal{A}\text{-alg}$ and called the *universal enveloping algebra functor*.

A triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a *good triple of operads* if $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ satisfies the following structure theorem : for any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H the following are equivalent :
1. The $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H is connected.
2. There is an isomorphism of connected coalgebras $H \cong \mathcal{C}^c(Prim(H))$.
3. There is an isomorphism of $\mathcal{C}^c$-$\mathcal{A}$-bialgebras $H \cong U(Prim(H))$.

We now give a rigidity theorem from [Foi10] concerning right graft algebras. Consider the deconcatenation coproduct $\Delta_{\mathcal{A}ss}$ on $\mathbf{H}_{NCK}$: for any forest $F \in \mathbf{H}_{NCK}$,

$$\Delta_{\mathcal{A}ss}(F) = \sum_{F_1, F_2 \in \mathbf{H}_{NCK}, F_1 F_2 = F} F_1 \otimes F_2.$$

Let $\tilde{\Delta}_{\mathcal{A}ss}$ be the coproduct on $\mathbf{M}_{NCK}$ obtained from $\Delta_{\mathcal{A}ss}$ by substracting its primitive part. We now have two products m and $\succ$ and one coproduct $\tilde{\Delta}_{\mathcal{A}ss}$ on $\mathbf{M}_{NCK}$.

Note that if A and B are two $\mathcal{RG}$-algebras then $A \overline{\otimes} B$ is a dipterous algebra (lemma 71). Recall that (see [Foi10]) an infinitesimal right graft bialgebra is a family $(A, *, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ where A is a $\mathbb{K}$-vector space, $*, \prec : A \otimes A \to A$ and $\tilde{\Delta}_{\mathcal{A}ss} : A \to A \otimes A$ are $\mathbb{K}$-linear maps, with the following compatibilities :

1. $(A, *, \prec)$ is a right graft algebra.

2. For all $x, y \in A$:

$$\left\{ \begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x * y) & = & (x \otimes 1) * \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x) * (1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array} \right. \tag{3.25}$$

It is proved in [Foi10] that $(\mathbf{M}_{NCK}, m, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ is an infinitesimal right graft bialgebra. In particular, with the first equality of (3.25), $(\mathbf{M}_{NCK}, \tilde{\Delta}_{\mathcal{A}ss})$ is an infinitesimal bialgebra (see [LR06]). If A is an infinitesimal right graft bialgebra, we note $Prim(A) = Ker(\tilde{\Delta}_{\mathcal{A}ss})$.

Let us recall that a magmatic algebra is a $\mathbb{K}$-vector space A equipped with a binary operation $\bullet$, without any relation. We do not suppose that magmatic algebras have units. The operad associated is denoted by $\mathcal{M}ag$.

L. Foissy prove in [Foi10] that for any infinitesimal right graft bialgebra, its primitive part is a $\mathcal{M}ag$-algebra. In particular, $Prim(\mathbf{M}_{NCK}) = \mathbb{K}(\mathbb{T}_{NCK})$ equipped with the product $\prec$ is a $\mathcal{M}ag$-algebra and it is the free $\mathcal{M}ag$-algebra generated by $\centerdot$.

Then we have the following important result (see [Foi10]) : The triple $(\mathcal{A}ss, \mathcal{RG}, \mathcal{M}ag)$ is a good triple of operads.

3.4.2 Infinitesimal bigraft bialgebras

In the section 3.3.3, we give the relationship between the coproduct $\Delta_{\mathbf{H}_{BG}}$ defined in the section 3.3.1 and the bigraft products (see theorem 74). These relationships do not permit to define a notion of bigraft bialgebra.

Consider an another coproduct on $\mathbf{H}_{BG}$, the deconcatenation coproduct $\Delta_{\mathcal{A}ss}$: for all $F \in \mathbb{F}_{BG}$,

$$\Delta_{\mathcal{A}ss}(F) = \sum_{F_1, F_2 \in \mathbb{F}_{BG}, F_1 F_2 = F} F_1 \otimes F_2.$$

We will show that, with this coproduct, we have good relationship with the bigraft products and we can define a notion of bigraft bialgebra.

We give in the following proposition another definition of $\succ$ and $\prec$, to be compared with lemma 73 (and we will use this definition in the following) :

Definition 75 *Let A and B be two bigraft algebras. Then we define three binary operations denoted by $\succ, \prec$ and $*$ on $A \overline{\otimes} B$ in the following way : for $a, a' \in A \cup \mathbb{K}$ and $b, b' \in B \cup \mathbb{K}$,*

$$\begin{array}{rcl} (a \otimes b) * (a' \otimes b') & = & (a * a') \otimes (b * b'), \\ (a \otimes b) \succ (a' \otimes b') & = & (a \succ a') \otimes (b * b'), \text{ if } a \text{ or } a' \in A, \\ (1 \otimes b) \succ (1 \otimes b') & = & 1 \otimes (b \succ b'), \\ (a \otimes b) \prec (a' \otimes b') & = & (a * a') \otimes (b \prec b'), \text{ if } b \text{ or } b' \in B, \\ (a \otimes 1) \prec (a' \otimes 1) & = & (a \prec a') \otimes 1. \end{array}$$

Remark. With lemma 71, $(A \overline{\otimes} B, *, \prec)$ is a right dipterous algebra. Applying the same reasoning to $(A \overline{\otimes} B, *, \succ)$, we prove that this is a left dipterous algebra. Moreover, the entanglement relation is not true in general : for $a, a'' \in A$ and $b, b', b'' \in B$,

$$\begin{array}{rcl} ((a \otimes b) \succ (1 \otimes b')) \prec (a'' \otimes b'') & = & ((a \succ 1) \otimes (b * b')) \prec (a'' \otimes b'') = 0 \\ (a \otimes b) \succ ((1 \otimes b') \prec (a'' \otimes b'')) & = & (a \succ a'') \otimes (b * (b' \prec b'')). \end{array}$$

We now have three products, namely m, $\succ$ and $\prec$ and one coproduct, namely $\tilde{\Delta}_{\mathcal{A}ss}$, on $\mathbf{M}_{BG}$, obtained from $\Delta_{\mathcal{A}ss}$ by substracting its primitive parts. We introduce the definition of infinitesimal bigraft bialgebra :

Definition 76 *An infinitesimal bigraft bialgebra is a family $(A, *, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ where $*, \succ, \prec : A \otimes A \to A$, $\tilde{\Delta}_{\mathcal{A}ss} : A \to A \otimes A$, with the following compatibilities :*

1. $(A, *, \succ, \prec)$ is a bigraft algebra.

2. For all $x, y \in A$:

$$\left\{\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x * y) & = & (x \otimes 1) * \tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x) * (1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array}\right. \tag{3.26}$$

Then the following properties illustrate the compatibility relationships :

Proposition 77 *$(\mathbf{M}_{BG}, m, \succ, \prec, \tilde{\Delta}_{\mathcal{A}ss})$ is an infinitesimal bigraft bialgebra. In particular, for all $x, y \in \mathbf{M}_{BG}$,*

$$\left\{\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(xy) & = & (x \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(y) + \tilde{\Delta}_{\mathcal{A}ss}(x)(1 \otimes y) + x \otimes y, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) & = & (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y), \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) & = & \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y). \end{array}\right. \tag{3.27}$$

Proof. With the proposition 45, $(\mathbf{M}_{BG}, m, \succ, \prec)$ is a bigraft algebra. It remains to prove the formulas (3.27). We can restrict ourselves to $F, G \in \mathbb{F}_{BG} \setminus \{1\}$. We put $F = F_1 \ldots F_n$, $G = G_1 \ldots G_m$ where the F_i's and the G_i's are trees and $G_1 = B_{BG}(G_1^1 \otimes G_1^2)$. Hence :

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(FG) & = & \displaystyle\sum_{H_1, H_2 \in \mathbb{F}_{BG} \setminus \{1\}, H_1 H_2 = FG} H_1 \otimes H_2 \\ & = & \displaystyle\sum_{H_1, H_2 \in \mathbb{F}_{BG} \setminus \{1\}, H_1 H_2 = G} FH_1 \otimes H_2 + \sum_{H_1, H_2 \in \mathbb{F}_{BG} \setminus \{1\}, H_1 H_2 = F} H_1 \otimes H_2 G + F \otimes G \\ & = & (F \otimes 1)\tilde{\Delta}_{\mathcal{A}ss}(G) + \tilde{\Delta}_{\mathcal{A}ss}(F)(1 \otimes G) + F \otimes G, \end{array}$$

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(F \succ G) & = & \tilde{\Delta}_{\mathcal{A}ss}(B_{BG}(FG_1^1 \otimes G_1^2)G_2 \ldots G_m) \\ & = & B_{BG}(FG_1^1 \otimes G_1^2) \otimes G_2 \ldots G_m + \displaystyle\sum_{i=2}^{m-1} B_{BG}(FG_1^1 \otimes G_1^2)G_2 \ldots G_i \otimes G_{i+1} \ldots G_m \\ & = & F \succ G_1 \otimes G_2 \ldots G_m + \displaystyle\sum_{i=2}^{m-1} F \succ G_1 G_2 \ldots G_i \otimes G_{i+1} \ldots G_m \\ & = & (F \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(G). \end{array}$$

Remark that $\Delta_{\mathcal{A}ss}(F^\dagger) = \tau \circ (\dagger \otimes \dagger) \circ \Delta_{\mathcal{A}ss}(F)$ for all $F \in \mathbb{F}_{BG}$. So we deduce the third relation of (3.27) from the second one in this way :

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(F \prec G) & = & \tilde{\Delta}_{\mathcal{A}ss}((G^\dagger \succ F^\dagger)^\dagger) \\ & = & \tau \circ (\dagger \otimes \dagger) \circ \tilde{\Delta}_{\mathcal{A}ss}(G^\dagger \succ F^\dagger) \\ & = & \tau \circ (\dagger \otimes \dagger)((G^\dagger \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger)) \\ & = & \tilde{\Delta}_{\mathcal{A}ss}(F^\dagger) \prec (1 \otimes G). \end{array}$$

□

Definition 78 *If A is an infinitesimal bigraft bialgebra, we note $Prim(A) = Ker(\tilde{\Delta}_{\mathcal{A}ss})$. In the infinitesimal bigraft bialgebra $\mathbf{M}_{BG}$, $Prim(\mathbf{M}_{BG}) = \mathbb{K}(\mathbb{T}_{BG})$ and we denote by $\mathbf{P}_{BG}$ the primitive part of $\mathbf{M}_{BG}$.*

Recall the definition of a $\mathcal{L}$-algebra introduced by P. Leroux in [Ler08] :

Definition 79 *A $\mathcal{L}$-algebra is a $\mathbb{K}$-vector space A equipped with two binary operations $\succ, \prec : A \otimes A \to A$ verifying the entanglement relation :*

$$(x \succ y) \prec z = x \succ (y \prec z),$$

for all $x, y, z \in A$.

The operad $\mathcal{L}$ is binary, quadratic, regular and set-theoretic. We denote by $\widetilde{\mathcal{L}}$ the nonsymmetric operad associated with the regular operad $\mathcal{L}$. We do not suppose that $\mathcal{L}$-algebras have unit for $\succ$ or $\prec$. If A and B are two $\mathcal{L}$-algebras, a $\mathcal{L}$-morphism from A to B is a $\mathbb{K}$-linear map $f : A \to B$ such that $f(x \succ y) = f(x) \succ f(y)$ and $f(x \prec y) = f(x) \prec f(y)$ for all $x, y \in A$. We denote by $\mathcal{L}$-alg the category of $\mathcal{L}$-algebras.

Proposition 80 *For any infinitesimal bigraft bialgebra, its primitive part is a $\mathcal{L}$-algebra.*

Proof. Let A be an infinitesimal bigraft bialgebra. We put $x, y \in Prim(A)$. Then, with (3.26),

$$\begin{array}{rcl} \tilde{\Delta}_{\mathcal{A}ss}(x \succ y) &=& (x \otimes 1) \succ \tilde{\Delta}_{\mathcal{A}ss}(y) = 0, \\ \tilde{\Delta}_{\mathcal{A}ss}(x \prec y) &=& \tilde{\Delta}_{\mathcal{A}ss}(x) \prec (1 \otimes y) = 0. \end{array}$$

Therefore, $x \succ y, x \prec y \in Prim(A)$. As we always have the relation $(x \succ y) \prec z = x \succ (y \prec z)$ for all $x, y, z \in Prim(A)$, $Prim(A)$ is a $\mathcal{L}$-algebra. $\square$

In particular, $(\mathbf{P}_{BG}, \succ, \prec)$ is a $\mathcal{L}$-algebra. We have even more than this :

Theorem 81 *$(\mathbf{P}_{BG}, \succ, \prec)$ is the free $\mathcal{L}$-algebra generated by $\bullet$.*

Proof. Let A be a $\mathcal{L}$-algebra and $a \in A$. Let us prove that there exists a unique morphism of $\mathcal{L}$-algebras $\phi : \mathbf{P}_{BG} \to A$ such that $\phi(\bullet) = a$. We define $\phi(F)$ for any nonempty tree $F \in \mathbf{P}_{BG}$ inductively on the degree of F by :

$$\left\{ \begin{array}{rcl} \phi(\bullet) &=& a, \\ \phi(F) &=& (\ldots((\phi(F_1^1) \succ (\ldots(\phi(F_{p-1}^1) \succ (\phi(F_p^1) \succ a))\ldots)) \prec \phi(F_1^2))\ldots) \prec \phi(F_q^2) \\ && \text{if } F = B_{BG}(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2) \text{ with the } F_i^1\text{'s and the } F_j^2\text{'s} \in \mathbb{T}_{BG}. \end{array} \right.$$

This map is linearly extended into a map $\phi : \mathbf{P}_{BG} \to A$. Let us show that it is a morphism of $\mathcal{L}$-algebras, that is to say $\phi(F \succ G) = \phi(F) \succ \phi(G)$ and $\phi(F \prec G) = \phi(F) \prec \phi(G)$ for all $F, G \in \mathbb{T}_{BG} \setminus \{1\}$. Note $F = B_{BG}(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2)$, $G = B_{BG}(G_1^1 \ldots G_r^1 \otimes G_1^2 \ldots G_s^2)$ with $F_1^1, \ldots, F_p^1, F_1^2, \ldots, F_q^2$ and $G_1^1, \ldots, G_r^1, G_1^2, \ldots, G_s^2$ in $\mathbb{T}_{BG}$. Then :

1. For $\phi(F \succ G) = \phi(F) \succ \phi(G)$,

$$\begin{array}{rcl} \phi(F \succ G) &=& \phi(B_{BG}(FG_1^1 \ldots G_r^1 \otimes G_1^2 \ldots G_s^2)) \\ &=& (\ldots((\phi(F) \succ (\phi(G_1^1) \succ (\ldots(\phi(G_r^1) \succ a)\ldots))) \prec \phi(G_1^2))\ldots) \prec \phi(G_s^2) \\ &=& \phi(F) \succ ((\ldots((\phi(G_1^1) \succ (\ldots(\phi(G_r^1) \succ a)\ldots)) \prec \phi(G_1^2))\ldots) \prec \phi(G_s^2)) \\ &=& \phi(F) \succ \phi(G). \end{array}$$

2. For $\phi(F \prec G) = \phi(F) \prec \phi(G)$,

$$\begin{array}{rcl} \phi(F \prec G) &=& \phi(B_{BG}(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2 G)) \\ &=& ((\ldots((\phi(F_1^1) \succ (\ldots(\phi(F_p^1) \succ a)\ldots)) \prec \phi(F_1^2))\ldots) \prec \phi(F_q^2)) \prec \phi(G) \\ &=& \phi(F) \prec \phi(G). \end{array}$$

So ϕ is a morphism of $\mathcal{L}$-algebras.

Let $\phi' : \mathbf{P}_{BG} \to A$ be another morphism of $\mathcal{L}$-algebras such that $\phi'(\bullet) = a$. For any forest $F_1^1 \ldots F_p^1$ and $F_1^2 \ldots F_q^2 \in \mathbb{F}_{BG}$,

$$\begin{array}{rl} & \phi'(B_{BG}(F_1^1 \ldots F_p^1 \otimes F_1^2 \ldots F_q^2)) \\ = & \phi'((\ldots((F_1^1 \succ (\ldots(F_{p-1}^1 \succ (F_p^1 \succ \bullet))\ldots)) \prec F_1^2)\ldots) \prec F_q^2) \\ = & (\ldots((\phi'(F_1^1) \succ (\ldots(\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ \phi'(\bullet)))\ldots)) \prec \phi'(F_1^2))\ldots) \prec \phi'(F_q^2) \\ = & (\ldots((\phi'(F_1^1) \succ (\ldots(\phi'(F_{p-1}^1) \succ (\phi'(F_p^1) \succ a))\ldots)) \prec \phi'(F_1^2))\ldots) \prec \phi'(F_q^2). \end{array}$$

So $\phi = \phi'$. $\square$

Remark. We deduce from theorem 81 that $dim\left(\widetilde{\mathcal{L}}(n)\right) = t_n^{\mathbf{H}_{BG}}$ for all $n \in \mathbb{N}^*$ (with the same reasoning as in corollary 49). We find again the result already given in [Ler08].

3.4.3 Rigidity theorem for infinitesimal bigraft bialgebras

The functor $(-)_{\mathcal{L}} : \{\mathcal{BG} - \text{alg}\} \to \{\mathcal{L} - \text{alg}\}$ associates to a $\mathcal{BG}$-algebra $(A, m, \succ, \prec)$ the $\mathcal{L}$-algebra $(A, \succ, \prec)$. Reciprocally, we define the universal enveloping bigraft algebra of a $\mathcal{L}$-algebra $(A, \succ, \prec)$ as follows :

Definition 82 *The universal enveloping bigraft algebra of a $\mathcal{L}$-algebra $(A, \succ, \prec)$, denoted by $U_{\mathcal{BG}}(A)$, is the augmentation ideal $\overline{T}(A)$ of the tensor algebra $T(A)$ over the $\mathbb{K}$-vector space A equipped with two operations also denoted by $\succ$ and $\prec$, and defined by : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \leq i \leq p$ and $1 \leq j \leq q$,*

$$\begin{array}{rcl} (a_1 \dots a_p) \succ (b_1 \dots b_q) & := & (a_1 \succ (\dots (a_{p-1} \succ (a_p \succ b_1)) \dots)) b_2 \dots b_q, \\ (a_1 \dots a_p) \prec (b_1 \dots b_q) & := & a_1 \dots a_{p-1}((\dots ((a_p \prec b_1) \prec b_2) \dots) \prec b_q). \end{array} \tag{3.28}$$

Then $(\overline{T}(A), m, \succ, \prec)$ is a nonunitary bigraft algebra, where m is the concatenation.

We denote by $\mathcal{L}(V)$ the free $\mathcal{L}$-algebra over a $\mathbb{K}$-vector space V. The functor $\mathcal{L}(-)$ is the left adjoint to the forgetful functor from $\mathcal{L}$-algebras to vector spaces. Because the operad $\mathcal{L}$ is regular, we get the following result :

Proposition 83 *Let V be a $\mathbb{K}$-vector space. Then the free $\mathcal{L}$-algebra on V is*

$$\mathcal{L}(V) = \bigoplus_{n \geq 1} \mathbb{K}\left(\mathbb{T}_{BG}(n)\right) \otimes V^{\otimes n},$$

equipped with the following binary operations : for all $F \in \mathbb{T}_{BG}(n)$, $G \in \mathbb{T}_{BG}(m)$, $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ and $w_1 \otimes \dots \otimes w_m \in V^{\otimes m}$,

$$\begin{array}{rcl} (F \otimes v_1 \otimes \dots \otimes v_n) \succ (G \otimes w_1 \otimes \dots \otimes w_m) & = & (F \succ G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m), \\ (F \otimes v_1 \otimes \dots \otimes v_n) \prec (G \otimes w_1 \otimes \dots \otimes w_m) & = & (F \prec G \otimes v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m). \end{array}$$

Proposition 84 *The universal enveloping bigraft algebra of the free $\mathcal{L}$-algebra is canonically isomorphic to the free bigraft algebra :*

$$U_{\mathcal{BG}}(\mathcal{L}(V)) \cong \mathcal{BG}(V).$$

Proof. Let us prove that the functor $U_{\mathcal{BG}} : \{\mathcal{L} - \text{alg}\} \to \{\mathcal{BG} - \text{alg}\}$ is the left adjoint to $(-)_{\mathcal{L}} : \{\mathcal{BG} - \text{alg}\} \to \{\mathcal{L} - \text{alg}\}$.

We put A a $\mathcal{L}$-algebra and B a $\mathcal{BG}$-algebra. Let $f : A \to (B)_{\mathcal{L}}$ be a morphism of $\mathcal{L}$-algebras. It determines uniquely a morphism of algebras $\tilde{f} : \overline{T}(A) \to B$ because (B, m) is an associative algebra. We endow $\overline{T}(A)$ with a $\mathcal{BG}$-algebra structure defined by (3.28). Then $\tilde{f} : U_{\mathcal{BG}}(A) \to B$ is a morphism of $\mathcal{BG}$-algebras : for all $p, q \in \mathbb{N}^*$ and $a_i, b_j \in A$, $1 \leq i \leq p$ and $1 \leq j \leq q$,

$$\begin{array}{rcl} \tilde{f}((a_1 \dots a_p) \succ (b_1 \dots b_q)) & = & \tilde{f}((a_1 \succ (\dots (a_{p-1} \succ (a_p \succ b_1)) \dots)) b_2 \dots b_q) \\ & = & (\tilde{f}(a_1) \succ (\dots (\tilde{f}(a_{p-1}) \succ (\tilde{f}(a_p) \succ \tilde{f}(b_1))) \dots)) \tilde{f}(b_2) \dots \tilde{f}(b_q) \\ & = & (\tilde{f}(a_1) \dots \tilde{f}(a_p)) \succ (\tilde{f}(b_1) \dots \tilde{f}(b_q)) \\ & = & \tilde{f}(a_1 \dots a_p) \succ \tilde{f}(b_1 \dots b_q), \end{array}$$

and

$$\begin{array}{rcl} \tilde{f}((a_1 \dots a_p) \prec (b_1 \dots b_q)) & = & \tilde{f}(a_1 \dots a_{p-1}((\dots ((a_p \prec b_1) \prec b_2) \dots) \prec b_q)) \\ & = & \tilde{f}(a_1) \dots \tilde{f}(a_{p-1})((\dots ((\tilde{f}(a_p) \prec \tilde{f}(b_1)) \prec \tilde{f}(b_2)) \dots) \prec \tilde{f}(b_q)) \\ & = & (\tilde{f}(a_1) \dots \tilde{f}(a_p)) \prec (\tilde{f}(b_1) \dots \tilde{f}(b_q)) \\ & = & \tilde{f}(a_1 \dots a_p) \prec \tilde{f}(b_1 \dots b_q). \end{array}$$

On the other hand, let $g : U_{\mathcal{BG}}(A) \to B$ be a morphism of $\mathcal{BG}$-algebras. From the construction of $U_{\mathcal{BG}}(A)$ it follows that the map $A \to U_{\mathcal{BG}}(A)$ is a $\mathcal{L}$-algebra morphism. Hence the composition $\tilde{g}$ with g gives a $\mathcal{L}$-algebra morphism $A \to B$.

These two constructions are inverse of each other, and therefore $U_{\mathcal{BG}}$ is the left adjoint to $(-)_{\mathcal{L}}$.

As $U_{\mathcal{BG}}$ is left adjoint to $(-)_{\mathcal{L}}$ and $\mathcal{L}(-)$ is left adjoint to the forgetful functor, the composite is the left adjoint to the forgetful functor from $\mathcal{BG}$-algebras to vector spaces. Hence it is the functor $\mathcal{BG}(-)$. □

Theorem 85 *For any infinitesimal bigraft bialgebra A over a field $\mathbb{K}$, the following are equivalent :*

1. *A is a connected infinitesimal bigraft bialgebra,*
2. *A is cofree among the connected coalgebras,*

3. A is isomorphic to $U_{\mathcal{BG}}(Prim(A))$ as an infinitesimal bigraft bialgebra.

Proof. We prove the following implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 1.
1. $\Rightarrow$ 2. If A is a connected infinitesimal bigraft bialgebra, then A is isomorphic to $(\overline{T}(Prim(A)), m, \Delta)$ as an infinitesimal bialgebra, where $\overline{T}(Prim(A))$ is the augmentation ideal of the tensor algebra over $Prim(A)$, m is the concatenation and Δ is the deconcatenation (see [LR06] for a proof). Therefore A is cofree.
2. $\Rightarrow$ 3. If A is cofree, then it is isomorphic as an infinitesimal bialgebra to $(\overline{T}(Prim(A)), m, \Delta)$ and $Prim(A)$ is a $\mathcal{L}$-algebra with proposition 80. $\overline{T}(Prim(A))$ is a $\mathcal{BG}$-algebra with the two operations $\succ$ and $\prec$ defined as in (3.28) and this is exactly $U_{\mathcal{BG}}(Prim(A))$. So A is isomorphic as an infinitesimal bialgebra to $U_{\mathcal{BG}}(Prim(A))$ and it is a $\mathcal{BG}$-morphism by using (3.7) and (3.28).
3. $\Rightarrow$ 1. By construction, $U_{\mathcal{BG}}(Prim(A))$ is isomorphic to $\overline{T}(Prim(A))$ as a bialgebra. Therefore A is isomorphic to $\overline{T}(Prim(A))$ as a bialgebra. As $\overline{T}(Prim(A))$ is connected, A is connected. $\square$

We deduce the following theorem :

Theorem 86 *The triple $(\mathcal{A}ss, \mathcal{BG}, \mathcal{L})$ is a good triple of operads.*

Remark. Note that if A is an infinitesimal bigraft bialgebra, then $(A, m, \tilde{\Delta}_{\mathcal{A}ss})$ is a nonunitary infinitesimal bialgebra. Hence, if $(\mathbb{K} \oplus A, m, \Delta_{\mathcal{A}ss})$ has an antipode S, then $-S$ is an eulerian idempotent for A and we have :

$$S(a) = \begin{cases} -a \text{ if } a \in Prim(A), \\ 0 \text{ if } a \in A^2. \end{cases}$$

Chapitre 4

Algèbres de greffes

Introduction

Dans ce chapitre, nous nous intéressons à différentes sous-algèbres de Hopf de $\mathbf{H}_o$, dont la construction a été inspirée par les travaux de F. Menous.

Dans [Men02], F. Menous étudie certains ensembles de probabilités, appelés moyennes induites par J. Ecalle, associés à une variable aléatoire sur $\mathbb{R}$. Une moyenne est un ensemble de "poids" indexés par des mots sur l'alphabet à deux éléments $\{+,-\}$. Pour un mot $(\varepsilon_1,\ldots,\varepsilon_n)$ donné ($\varepsilon_i=\pm$), le poids est simplement la probabilité d'appartenir à $\mathbb{R}^{\varepsilon_1}$ au temps 1, à $\mathbb{R}^{\varepsilon_2}$ au temps 2, ..., à $\mathbb{R}^{\varepsilon_n}$ au temps n. F. Menous prouve dans [Men02] qu'un tel coefficient peut être décomposé en une somme de coefficients élémentaires qui sont indexés par des arbres et des forêts ordonnés. Pour cela, il construit par récurence, avec un formalisme proche de celui du calcul moulien, un ensemble de forêts ordonnées. C'est cet ensemble, noté $\mathbb{G}$, qu'on se propose d'étudier ici.

Pour cela, nous définissons deux opérateurs de greffes B^+ et B^-, à partir desquels nous proposons une construction de l'ensemble $\mathbb{G}$, ce qui permet de définir l'algèbre $\mathbf{B}^\infty=\mathbb{K}[\mathbb{G}]$. Une étude combinatoire de $\mathbb{G}$ permet de démontrer que $\mathbf{B}^\infty$ est une algèbre de Hopf et de calculer sa série formelle.

Toujours avec les opérateurs B^+ et B^-, on construit une suite croissante $\left(\mathbb{G}^i\right)_{i\geq 1}$ de sous-ensembles de $\mathbb{G}$ ce qui permet de définir des algèbres $\mathbf{B}^i=\mathbb{K}[\mathbb{G}^i]$. Nous démontrons que pour tout $i\geq 1$, $\mathbf{B}^i$ est une algèbre de Hopf et on obtient ainsi un "dévissage" de l'algèbre de Hopf $\mathbf{B}^\infty$ avec les inclusions $\mathbf{B}^1\subseteq\ldots\subseteq\mathbf{B}^i\subseteq\mathbf{B}^{i+1}\subseteq\ldots\subseteq\mathbf{B}^\infty$.

En généralisant la construction de F. Menous, nous définissons une algèbre $\mathbf{B}=\mathbb{K}[\mathbb{T}]$ à partir d'un ensemble d'arbres ordonnés $\mathbb{T}$ construit avec les opérateurs B^+ et B^-. On démontre que $\mathbf{B}$ est une algèbre de Hopf. On munit $\mathbf{B}$ d'une structure de bialgèbre dupliciale dendriforme (voir [Foi07, Foi12, Lod08]) ce qui permet, par des raisonnements proches de ceux utilisés dans [Foi12], de montrer sa coliberté et son auto-dualité.

On s'attache enfin à définir sur $\mathbf{B}$ une greffe à gauche et une greffe à droite de sorte que $\mathbf{B}$ est une algèbre bigreffe et on démontre qu'elle est engendrée comme algèbre bigreffe par l'unique arbre de degré 1.

Le chapitre est organisé comme suit : dans la première partie, nous construisons, avec des opérateurs de greffes B^+ et B^-, un ensemble $\mathbb{G}$ d'arbres ordonnés et nous exhibons différentes propriétés combinatoires de $\mathbb{G}$. À partir de cet ensemble, nous définissons l'algèbre $\mathbf{B}^\infty$, nous démontrons que c'est une algèbre de Hopf et nous calculons sa série formelle. On construit ensuite des sous-algèbres $\mathbf{B}^i$ de $\mathbf{B}^\infty$, pour $i\in\mathbb{N}^*$, et on démontre que ce sont des algèbres de Hopf. Dans la seconde partie, nous définissons l'algèbre $\mathbf{B}$ à partir d'un ensemble $\mathbb{T}$ lui aussi construit avec les opérateurs de greffes B^+ et B^-. On munit $\mathbf{B}$ d'une structure de bialgèbre dupliciale dendriforme et cela nous permet de montrer la coliberté et l'auto-dualité de $\mathbf{B}$. Enfin, on munit $\mathbf{B}$ d'une structure d'algèbre bigreffe et on démontre que $\mathbf{B}$ est engendrée comme algèbre bigreffe par l'élément $\bullet_1$.

Les résultats de ce chapitre sont rassemblés dans un article intitulé *Algèbres de greffes* publié dans Bulletin des Sciences Mathématiques 136 (2012), no.8, 904-939.

Remarque. Dans ce chapitre, le degré d'un arbre ou d'une forêt est le degré en sommets.

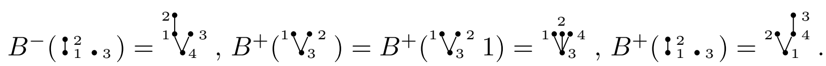
4.1 Les algèbres de Hopf $\mathbf{B}^i$

4.1.1 Construction et étude de $\mathbb{G}$

Commençons par introduire deux opérateurs de greffes B^+ et B^- qui sont utilisés dans tout ce qui suit. Pour cela, considèrons une suite de m arbres ordonnés non vides $T_1, \ldots, T_m$ dont la somme des degrés est notée n. On pose :

1. $B^-(T_1 \ldots T_m)$ l'arbre ordonné de degré $n+1$ obtenu comme suit : on considère $T_1, \ldots, T_m$ comme la suite des sous-arbres d'un arbre enraciné ayant pour racine le sommet indexé par $n+1$. De plus, on conviendra que $B^-(1)$ est égale à l'arbre $\bullet_1$.
2. $B^+(T_1 \ldots T_m)$ l'arbre ordonné de degré $n+1$ construit en greffant le sommet indexé par $n+1$ comme le fils le plus à droite de la racine de T_1 et en considérant alors $T_2, \ldots, T_m$ comme la suite des sous-arbres issus du sommet indexé par $n+1$. En particulier, on notera $B^+(T_1) = B^+(T_1 1)$ l'arbre obtenu en greffant le sommet indexé par $|T_1|_v + 1$ comme le fils le plus à droite de la racine de T_1. De plus, on conviendra que $B^+(1)$ est égale à l'arbre $\bullet_1$.

Note. Les opérateurs B^+ et B^- sont différents de ceux introduits dans [CK98] par A. Connes et D. Kreimer.

Exemples. 

Pour toute suite $\underline{\varepsilon} = \varepsilon_1, \ldots, \varepsilon_n \in \{+,-\}^n$, avec $n \geq 1$, on définit par récurrence un ensemble $\mathbb{G}^{(\underline{\varepsilon})}$ qui correspond à un ensemble de forêts ordonnées de degré n.

Si $\underline{n=1}$, $\mathbb{G}^{(\varepsilon_1)}$, pour ε_1 quelconque, est l'ensemble réduit à un seul élément, la forêt de degré 1, l'unique sommet étant évidemment indexé par 1.

Si $\underline{n \geq 2}$, considérons l'ensemble $\mathbb{G}^{(\varepsilon_1, \ldots, \varepsilon_{n-1})}$ déjà construit.

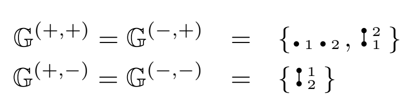
1. Si $\varepsilon_n = -$, les éléments F de $\mathbb{G}^{(\varepsilon_1, \ldots, \varepsilon_n)}$ sont obtenus par la transformation suivante. On prend un élément $F' = T_1 \ldots T_m$ de $\mathbb{G}^{(\varepsilon_1, \ldots, \varepsilon_{n-1})}$, avec $m \geq 1$, et on considère $T_1, \ldots, T_m$ comme la suite des sous-arbres d'un arbre enraciné ayant pour racine le sommet indexé par n. Cela donne ainsi naissance à une nouvelle forêt ordonnée de degré n, avec un seul arbre, égale à $B^-(T_1 \ldots T_m)$.
2. Si $\varepsilon_n = +$, alors comme précédement on considère un élément F' de $\mathbb{G}^{(\varepsilon_1, \ldots, \varepsilon_{n-1})}$. Nous avons alors plusieurs possibilités pour ajouter un nouveau sommet indexé par n. Notons encore $F' = T_1 \ldots T_m$, où $T_1, \ldots, T_m$ est la suite des arbres qui composent la forêt F' et $m \geq 1$. On peut alors faire l'une des transformations suivantes pour obtenir un élément de $\mathbb{G}^{(\varepsilon_1, \ldots, \varepsilon_n)}$.
   (a) Concaténer l'unique arbre de degré 1 indexé par n à la forêt $T_1 \ldots T_m$ sur la droite. Cela donne la forêt ordonnée de degré n égale à $T_1 \ldots T_m \bullet_n$.
   (b) Pour $1 \leq i \leq m$, greffer le sommet indexé par n comme le fils le plus à droite de la racine de T_i et considérer alors $T_{i+1}, \ldots, T_m$ comme la suite des sous-arbres issus du sommet indexé par n. On obtient alors la forêt ordonnée de degré n égale à $T_1 \ldots T_{i-1} B^+(T_i \ldots T_m)$.

Voici une illustration de cette construction :

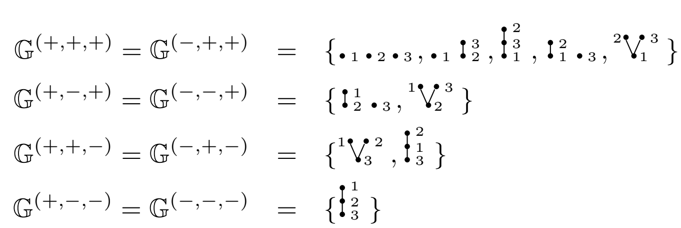
– Pour $n = 1$:

$$\mathbb{G}^{(+)} = \mathbb{G}^{(-)} = \{\bullet_1\}$$

– Pour $n = 2$:

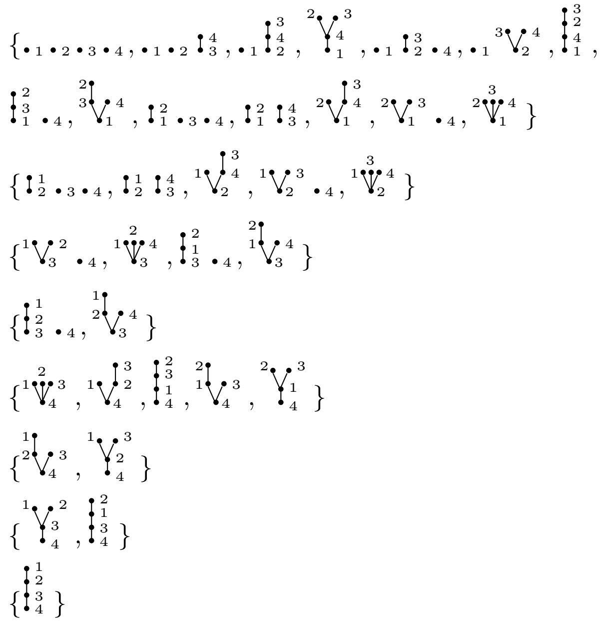


– Pour $n = 3$:



– Pour $n = 4$:

$$\mathbb{G}^{(+,+,+,+)} = \mathbb{G}^{(-,+,+,+)} =$$
$$\mathbb{G}^{(+,-,+,+)} = \mathbb{G}^{(-,-,+,+)} =$$
$$\mathbb{G}^{(+,+,-,+)} = \mathbb{G}^{(-,+,-,+)} =$$
$$\mathbb{G}^{(+,-,-,+)} = \mathbb{G}^{(-,-,-,+)} =$$
$$\mathbb{G}^{(+,+,+,-)} = \mathbb{G}^{(-,+,+,-)} =$$
$$\mathbb{G}^{(+,-,+,-)} = \mathbb{G}^{(-,-,+,-)} =$$
$$\mathbb{G}^{(+,+,-,-)} = \mathbb{G}^{(-,+,-,-)} =$$
$$\mathbb{G}^{(+,-,-,-)} = \mathbb{G}^{(-,-,-,-)} =$$



Dans la suite, étant donné $\underline{\varepsilon} \in \{+,-\}^n$ avec $n \geq 1$, on identifiera toujours les deux ensembles $\mathbb{G}^{(+,\underline{\varepsilon})}$ et $\mathbb{G}^{(-,\underline{\varepsilon})}$ (et les deux ensembles $\mathbb{G}^{(+)}$ et $\mathbb{G}^{(-)}$). On préferera, suivant les cas, dire qu'une forêt appartient à $\mathbb{G}^{(+,\underline{\varepsilon})}$ ou à $\mathbb{G}^{(-,\underline{\varepsilon})}$.

Pour $n \geq 2$, considérons un élément F de $\mathbb{G}^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}$. Notons $S_{\varepsilon_n}(F)$ l'ensemble des forêts construites à partir de F par les méthodes de construction ci-dessus, suivant la valeur de ε_n. Alors,

$$\mathbb{G}^{(\varepsilon_1,\ldots,\varepsilon_n)} = \bigcup_{F \in \mathbb{G}^{(\varepsilon_1,\ldots,\varepsilon_{n-1})}} S_{\varepsilon_n}(F). \tag{4.1}$$

Le lemme suivant sera utile dans la suite :

Lemme 87 *Soient $\underline{\varepsilon}, \underline{\varepsilon}' \in \bigcup_{n\geq 1} \{+,-\}^n$, avec $\underline{\varepsilon} \neq \underline{\varepsilon}'$. Alors $\mathbb{G}^{(+,\underline{\varepsilon})} \cap \mathbb{G}^{(+,\underline{\varepsilon}')} = \emptyset$.*

Preuve. Soient $\underline{\varepsilon}, \underline{\varepsilon}' \in \cup_{n\geq 1} \{+,-\}^n$, avec $\underline{\varepsilon} \neq \underline{\varepsilon}'$. Tout d'abord, supposons que la suite $\underline{\varepsilon}$ est de longueur n et que la suite $\underline{\varepsilon}'$ est de longueur n', avec $n \neq n'$. Comme les éléments de $\mathbb{G}^{(+,\underline{\varepsilon})}$ sont de degré $n+1$ et ceux de $\mathbb{G}^{(+,\underline{\varepsilon}')}$ sont de degré $n'+1$, $\mathbb{G}^{(+,\underline{\varepsilon})} \cap \mathbb{G}^{(+,\underline{\varepsilon}')} = \emptyset$.

Supposons maintenant que les suites $\underline{\varepsilon}$ et $\underline{\varepsilon}'$ sont de même longueur $n \geq 1$ et raisonnons par récurrence sur n. Le résultat est trivial pour $n = 1$. Supposons $n \geq 2$. On distingue alors deux cas :

1. Si $\varepsilon_n \neq \varepsilon'_n$, par exemple $\varepsilon_n = -$ et $\varepsilon'_n = +$. Les éléments de $\mathbb{G}^{(+,\underline{\varepsilon})}$ sont tous des arbres dont la racine est indexée par $n+1$. L'ensemble $\mathbb{G}^{(+,\underline{\varepsilon}')}$ est constitué d'arbres et de forêts (de longueur ≥ 2). Par construction, les arbres de $\mathbb{G}^{(+,\underline{\varepsilon}')}$ sont de la forme $B^+(T_1 \ldots T_m)$, avec $m \geq 1$ et $T_1 \ldots T_m \in \mathbb{G}^{(+,\varepsilon'_1,\ldots,\varepsilon'_{n-1})}$. En particulier, la racine de ces arbres est indexée par un entier $< n+1$. Donc $\mathbb{G}^{(+,\underline{\varepsilon})} \cap \mathbb{G}^{(+,\underline{\varepsilon}')} = \emptyset$.

2. Si $\varepsilon_n = \varepsilon'_n$, comme $\underline{\varepsilon} \neq \underline{\varepsilon}'$, $\varepsilon_1,\ldots,\varepsilon_{n-1} \neq \varepsilon'_1,\ldots,\varepsilon'_{n-1}$. Par l'absurde, supposons que $\mathbb{G}^{(+,\underline{\varepsilon})} \cap \mathbb{G}^{(+,\underline{\varepsilon}')} \neq \emptyset$. Alors il existe une forêt $T_1 \ldots T_m \in \mathbb{G}^{(+,\varepsilon_1,\ldots,\varepsilon_{n-1})}$ et une forêt $T'_1 \ldots T'_l \in \mathbb{G}^{(+,\varepsilon'_1,\ldots,\varepsilon'_{n-1})}$ telles que $S_{\varepsilon_n}(T_1 \ldots T_m) \cap S_{\varepsilon'_n}(T'_1 \ldots T'_l) \neq \emptyset$. En utilisant l'hypothèse de récurrence, $\mathbb{G}^{(+,\varepsilon_1,\ldots,\varepsilon_{n-1})} \cap \mathbb{G}^{(+,\varepsilon'_1,\ldots,\varepsilon'_{n-1})} = \emptyset$, donc $T_1 \ldots T_m \neq T'_1 \ldots T'_l$. Or

   (a) si $\varepsilon_n = \varepsilon'_n = -$, $S_-(T_1 \ldots T_m) = \{B^-(T_1 \ldots T_m)\}$, $S_-(T'_1 \ldots T'_l) = \{B^-(T'_1 \ldots T'_l)\}$. Donc $B^-(T_1 \ldots T_m) = B^-(T'_1 \ldots T'_l)$ et, nécessairement, $m = l$ et $T_1 \ldots T_m = T'_1 \ldots T'_l$. On aboutit donc à une contradiction.

(b) si $\varepsilon_n = \varepsilon'_n = +$, alors

$$\begin{array}{rcl} S_+(T_1 \dots T_m) & = & \{B^+(T_1 \dots T_m), T_1 B^+(T_2 \dots T_m), \dots, T_1 \dots T_{m-1} B^+(T_m), \\ & & T_1 \dots T_m \bullet_{n+1}\} \\ S_+(T'_1 \dots T'_l) & = & \{B^+(T'_1 \dots T'_l), T'_1 B^+(T'_2 \dots T'_l), \dots, T'_1 \dots T'_{l-1} B^+(T'_l), \\ & & T'_1 \dots T'_l \bullet_{n+1}\}. \end{array}$$

Alors il existe $i \in \{1, \dots m\}$ et $j \in \{1, \dots l\}$ tels que on a l'égalité $T_1 \dots T_{i-1} B^+(T_i \dots T_m) = T'_1 \dots T'_{j-1} B^+(T'_j \dots T'_l)$. Nécessairement, on doit avoir $i = j$, $T_1 = T'_1, \dots, T_{i-1} = T'_{i-1}$ et $B^+(T_i \dots T_m) = B^+(T'_i \dots T'_l)$. Or cette dernière égalité implique que $m = l$ et $T_i = T'_i, \dots, T_m = T'_m$. Ici encore, cela contredit $T_1 \dots T_m \neq T'_1 \dots T'_l$.

Par récurrence, le résultat est ainsi démontré. $\square$

En reprenant la preuve précédente, remarquons que dans l'égalité (4.1) l'union est disjointe. De plus, dans chaque ensemble $S_{\varepsilon_n}(F)$, toutes les forêts sont distinctes car elles ne sont pas de même longueur. Ainsi, pour tout $\underline{\varepsilon} \in \{+,-\}^n$, avec $n \geq 1$, les éléments de $\mathbb{G}^{(\underline{\varepsilon})}$ sont tous distincts.

Considérons les ensembles suivants :

$$\mathbb{G} = \bigcup_{\underline{\varepsilon} \in \{+,-\}^n, n \geq 1} \mathbb{G}^{(\underline{\varepsilon})} \text{ et } \mathbb{G}^0 = \bigcup_{n \geq 1} \mathbb{G}^{(\overbrace{+, \dots, +}^{n \text{ fois}})}.$$

D'après le lemme 87, les deux unions précédentes sont disjointes (à l'identification près des ensembles $\mathbb{G}^{(+,\underline{\varepsilon})}$ et $\mathbb{G}^{(-,\underline{\varepsilon})}$) et il n'y a donc pas de redondances dans la construction des forêts appartenant à $\mathbb{G}$.

Remarquons que $\mathbb{G}$ n'est pas stable pour l'opération de concaténation. Par exemple, les arbres $\mathfrak{l}^{2}_{1}$ et $\mathfrak{l}^{1}_{2}$ appartiennent à $\mathbb{G}$ mais la forêt $\mathfrak{l}^{2}_{1} \mathfrak{l}^{1}_{2} \notin \mathbb{G}$. Cela est dû au fait que l'arbre de droite qui compose la forêt $\mathfrak{l}^{2}_{1} \mathfrak{l}^{1}_{2}$ a été construit avec l'opérateur B^- ($\mathfrak{l}^{1}_{2} \in \mathbb{G}^{(+,-)}$). Plus précisément, on a le résultat suivant :

Lemme 88 *Les assertions suivantes sont équivalentes :*

1. *La forêt $T_1 \dots T_m$ appartient à $\mathbb{G}$.*
2. *$T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$.*

En particulier, pour toute forêt $T_1 \dots T_m \in \mathbb{G}$, $T_1, \dots, T_m$ appartiennent à $\mathbb{G}$.

Preuve. Démontrons tout d'abord le sens direct. Soit une forêt $T_1 \dots T_m$ appartenant à $\mathbb{G}$ et notons $n = |T_1|_v + \dots + |T_m|_v$ son degré. D'après le lemme 87, il existe un unique $\underline{\varepsilon} \in \{+,-\}^n$ tel que $T_1 \dots T_m \in \mathbb{G}^{(\underline{\varepsilon})}$. Soit i le plus grand indice tel que $\varepsilon_i = -$. Par construction, les éléments appartenant à $\mathbb{G}^{(\varepsilon_1, \dots, \varepsilon_i)}$ sont des arbres de la forme B^- de l'arbre vide si $i = 1$ ou B^- d'une suite d'arbres $G_1, \dots, G_k$ telle que $G_1 \dots G_k \in \mathbb{G}^{(\varepsilon_1, \dots, \varepsilon_{i-1})}$ si $i \geq 2$. T_1 contient donc un sous-arbre $B^-(G_1 \dots G_k)$ ($B^-(1)$ si $i = 1$) appartenant à $\mathbb{G}^{(\varepsilon_1, \dots, \varepsilon_i)}$. Autrement dit, $\exists\, l \geq 0$ tel que

$$T_1 = \overbrace{B^+(\dots B^+}^{l \text{ fois}}(B^-(G_1 \dots G_k) \dots) \dots).$$

Ainsi, $|T_1|_v \geq i$. Par ailleurs, remarquons que si $F_1 F_2 \in \mathbb{G}^{(\underline{\varepsilon})}$, alors $F_1 \in \mathbb{G}^{(\varepsilon_1, \dots, \varepsilon_{|F_1|_v})}$ et $F_2 \in \mathbb{G}^{(\varepsilon_{|F_1|_v+1}, \dots, \varepsilon_{|F_1|_v + |F_2|_v})}$. Ainsi, $T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$.

Réciproquement, montrons que si $T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$, alors $T_1 T_2 \dots T_m \in \mathbb{G}$ par récurrence sur $|F|_v$, où $F = T_2 \dots T_m$. Si $|F|_v = 1$, c'est-à-dire $F = \bullet_1$, alors $T_1 F = T_1 \bullet_{|T_1|_v + 1}$, et par construction de $\mathbb{G}$, $T_1 F \in \mathbb{G}$. Soit $n \geq 1$ et supposons le résultat vérifié pour tout F tel que $|F|_v \leq n$. Considérons $F = T_2 \dots T_m$ de degré $n+1$, avec $T_2, \dots, T_m \in \mathbb{G}^0$. Si $|T_m|_v = 1$, par hypothèse de récurence $T_1 T_2 \dots T_{m-1} \in \mathbb{G}$ et comme pour l'initialisation $T_1 T_2 \dots T_{m-1} \bullet_{|T_1|_v + n + 1} \in \mathbb{G}$. Sinon, comme $T_m \in \mathbb{G}^0$, il existe $G_1, \dots, G_k \in \mathbb{G}^0$ tel que $T_m = B^+(G_1 \dots G_k)$, avec $G_1 \dots G_k$ une foret de degré $|T_m|_v - 1$. Par hypothèse de récurrence, $T_1 \dots T_{m-1} G_1 \dots G_k \in \mathbb{G}$, donc $T_1 F = T_1 T_2 \dots T_{m-1} B^+(G_1 \dots G_k)$ appartient bien à $\mathbb{G}$. $\square$

D'après le lemme précédent, $\mathbb{G}$ est stable par concaténation à gauche par des éléments de $\mathbb{G}^0$. Pour $\mathbb{G}^0$, on a le

Lemme 89 $\mathbb{G}^0 \cup \{1\}$ *est un monoïde libre pour l'opération de concaténation.*

Preuve. En effet, si $T_1T_2 \in \mathbb{G}^0$, $T_1T_2 \in \mathbb{G}^{\overbrace{(+,\ldots,+)}^{|T_1|_v+|T_2|_v \text{ fois}}}$, donc $T_1 \in \mathbb{G}^{\overbrace{(+,\ldots,+)}^{|T_1|_v \text{ fois}}} \subseteq \mathbb{G}^0$ et $T_2 \in \mathbb{G}^{\overbrace{(+,\ldots,+)}^{|T_2|_v \text{ fois}}} \subseteq \mathbb{G}^0$.

Réciproquement, supposons que $T_1, T_2 \in \mathbb{G}^0$ et raisonnons par récurrence sur le degré de T_2. Si $|T_2|_v = 1$, $T_2 = \bullet_1$ et alors $T_1T_2 \in \mathbb{G}^0$ par construction de $\mathbb{G}^0$. Supposons $|T_2|_v \geq 2$. Comme $T_2 \in \mathbb{G}^0$, il existe $G_1, \ldots, G_k \in \mathbb{G}^0$ tels que $T_2 = B^+(G_1 \ldots G_k)$. Alors la forêt $T_1G_1 \ldots G_k \in \mathbb{G}^0$ par hypothèse de récurrence, et donc $T_1T_2 = T_1B^+(G_1 \ldots G_k) \in \mathbb{G}^0$, par construction de $\mathbb{G}^0$. $\square$

Remarque. Posons $\Delta^{\boldsymbol{l}}_{\mathbf{H}_o}(F) = \displaystyle\sum_{\boldsymbol{v} \models V(F) \text{ et } R^{\boldsymbol{l}}_F \in Roo_{\boldsymbol{v}}(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F)$ pour toute forêt non vide F, où $R^{\boldsymbol{l}}_F$ est la racine de l'arbre le plus à gauche de la forêt F. Soient $T_1, \ldots, T_m$ m arbres non vides, $m \geq 1$. Alors, en étudiant les coupes admissibles :

$$\begin{array}{rcl}
\Delta_{\mathbf{H}_o}(B^-(T_1 \ldots T_m)) & = & (Id \otimes B^-) \circ \Delta_{\mathbf{H}_o}(T_1 \ldots T_m) + B^-(T_1 \ldots T_m) \otimes 1, \\
\Delta_{\mathbf{H}_o}(B^+(T_1 \ldots T_m)) & = & (Id \otimes B^+) \circ \Delta^{\boldsymbol{l}}_{\mathbf{H}_o}(T_1 \ldots T_m) + B^+(T_1 \ldots T_m) \otimes 1 \\
& & + \Delta^{\boldsymbol{l}}_{\mathbf{H}_o}(T_1) \cdot (B^-(T_2 \ldots T_m) \otimes 1).
\end{array}$$

Lemme 90 *Soit T un arbre appartenant à $\mathbb{G}^0 \cup \{1\}$. Alors, pour toute coupe admissible $\boldsymbol{v} \models V(T)$, $Roo_{\boldsymbol{v}}(T) \in \mathbb{G}^0 \cup \{1\}$.*

Preuve. Il suffit de montrer que, pour toute coupe simple $\boldsymbol{v} \models V(T)$ et pour tout arbre $T \in \mathbb{G}^0 \cup \{1\}$, $Roo_{\boldsymbol{v}}(T) \in \mathbb{G}^0 \cup \{1\}$. On raisonne par récurrence sur le degré n de $T \in \mathbb{G}^0 \cup \{1\}$, le résultat étant trivial si $n = 0, 1, 2$.

Supposons $n \geq 3$. Comme $T \in \mathbb{G}^0$, il existe $m \geq 1$, $T_1, \ldots, T_m \in \mathbb{G}^0$, tels que $T = B^+(T_1 \ldots T_m)$. Soit $\boldsymbol{v} \models V(T)$ une coupe simple de T. Il y a trois cas possibles :

1. Si $\boldsymbol{v} \models V(T_1)$, par hypothèse de récurrence, $Roo_{\boldsymbol{v}}(T_1) \in \mathbb{G}^0 \cup \{1\}$. Si $Roo_{\boldsymbol{v}}(T_1) = 1$, $Roo_{\boldsymbol{v}}(T) = 1$. Si $Roo_{\boldsymbol{v}}(T_1) \neq 1$, alors, avec le lemme 89, $Roo_{\boldsymbol{v}}(T_1)T_2 \ldots T_m \in \mathbb{G}^0$. Ainsi on a donc $Roo_{\boldsymbol{v}}(T) = B^+(Roo_{\boldsymbol{v}}(T_1)T_2 \ldots T_m) \in \mathbb{G}^0$.
2. Si $\boldsymbol{v} \models V(T_i)$, avec $i \geq 2$ (on inclut ici le cas de la coupe totale qui correspond à couper l'arête entre le sommet indexé par n et la racine de T_i). Par hypothèse de récurrence, $Roo_{\boldsymbol{v}}(T_i) \in \mathbb{G}^0 \cup \{1\}$. Avec le lemme 89, $T_1 \ldots Roo_{\boldsymbol{v}}(T_i) \ldots T_m \in \mathbb{G}^0$, et ainsi $Roo_{\boldsymbol{v}}(T) = B^+(T_1 \ldots Roo_{\boldsymbol{v}}(T_i) \ldots T_m) \in \mathbb{G}^0$.
3. Enfin, si on coupe l'arête joignant la racine de T (qui est aussi la racine de T_1) et le sommet indexé par n, alors $Roo_{\boldsymbol{v}}(T) = T_1 \in \mathbb{G}^0$.

Ainsi, dans tous les cas, $Roo_{\boldsymbol{v}}(T) \in \mathbb{G}^0$, et on peut conclure par le principe de récurrence. $\square$

Remarque. Par contre, étant donné un arbre T appartenant à $\mathbb{G}^0$, il existe certaines coupes $\boldsymbol{v} \models V(T)$ telles que $Lea_{\boldsymbol{v}}(T) \notin \mathbb{G}^0 \cup \{1\}$. Par exemple, considérons $T_1, \ldots, T_m \in \mathbb{G}^0$, avec $m \geq 2$, et $T = B^+(T_1 \ldots T_m) \in \mathbb{G}^0$. Alors, si on réalise la coupe simple $\boldsymbol{v} \models V(T)$ consistant à couper l'arête joignant la racine de T (qui est aussi la racine de T_1) et le sommet indexé par $|T|_v$, $Lea_{\boldsymbol{v}}(T) = B^-(T_2 \ldots T_m) \in \mathbb{G}^{(\ldots,-)}$ et donc $\notin \mathbb{G}^0$ en utilisant le lemme 87.

Soit T un arbre enraciné plan non vide $\in \mathbf{H}_{NCK}$. Une question naturelle est de savoir de combien de façons on peut indexer les sommets de T pour obtenir un élément de $\mathbb{G}^0$ ou de $\mathbb{G}$.

Dans le cas de $\mathbb{G}^0$, il n'y a qu'une seule et unique indexation possible. En effet, considérons un arbre plan T. Si T est de degré 1, le résultat est trivial. Sinon, $T = B_{NCK}(T_1 \ldots T_m)$ avec $T_1, \ldots, T_m$ m arbres non vides $\in \mathbf{H}_{NCK}$ et $m \geq 1$. Par hypothèse de récurrence, il y a une unique indexation de $B_{NCK}(T_1 \ldots T_{m-1})$ en un élément noté G de $\mathbb{G}^0$. Si $T_m = \bullet$, comme le descendant le plus à droite de la racine de T doit nécessairement être indexé par $|T|_v$, $B^+(G)$ est l'unique arbre de $\mathbb{G}^0$ tel que l'arbre plan associé (en supprimant l'indexation) soit T. Sinon, $T_m = B_{NCK}(T_{m,1} \ldots T_{m,k})$. Avec l'hypothèse de récurrence, $\forall\ 1 \leq i \leq k$, l'arbre plan $T_{m,i}$ a une unique indexation en un élément noté G_i de $\mathbb{G}^0$. Alors,

$B^+(GG_1 \dots G_k) \in \mathbb{G}^0$, et c'est par construction l'unique arbre de $\mathbb{G}^0$ tel que l'arbre plan associé soit T. On définit ainsi une bijection entre les arbres de $\mathbf{H}_{NCK}$ et les arbres de $\mathbb{G}^0$.

Pour le cas plus complexe de $\mathbb{G}$, on a la

Proposition 91 *Soit un arbre plan $T \in \mathbf{H}_{NCK}$. Rappelons que, si v est un sommet de T, $f(v)$ désigne la fertilité de v et $h(v)$ la hauteur de v. Notons L_T^l la feuille la plus à gauche de T. Alors, il y a*

$$1 + \sum_{i=0}^{h(L_T^l)-1} \prod_{L_T^l \twoheadrightarrow v \text{ et } h(v) \leq i} f(v)$$

façons d'indexer T pour obtenir un élément de $\mathbb{G}$.

Preuve. Rappelons que si T est un arbre non vide de $\mathbb{G}$, $B^+(T1) = B^+(T)$ est l'arbre construit en greffant le sommet indexé par $|T|_v + 1$ comme le fils le plus à droite de la racine de T. En particulier, on ne cherchera pas dans cette preuve à simplifier un produit d'arbres en supprimant les éventuels arbres vides.

Raisonnons par récurrence sur $k = h(L_T^l)$ la hauteur de L_T^l. Pour éviter d'alourdir les notations, on notera indifférement un arbre appartenant à $\mathbf{H}_{NCK}$ et l'arbre (ou les arbres) obtenu après indexation des sommets appartenant à $\mathbb{G}$.

Si $k = 0$, T est l'arbre plan réduit à sa racine, il y a donc une unique façon de le numéroter, en $\bullet_1$. Ceci démontre la formule au rang 0.

Si $k = 1$, $T = B_{NCK}(\bullet T_1 \dots T_n)$, avec $T_1 = B_{NCK}(T_{1,1} \dots T_{1,m_1}), \dots, T_n = B_{NCK}(T_{n,1} \dots T_{n,m_n})$ (si $T_i = \bullet$, $T_i = B_{NCK}(T_{i,1})$ avec $T_{i,1} = 1$). Il y a alors deux cas possibles pour l'indexation de T :

1. Si l'indice de la racine de T est plus petit que l'indice de son descendant direct le plus à gauche. Alors,
$$T = B^+(\dots B^+(B^+(\bullet_1 1) T_{1,1} \dots T_{1,m_1}) \dots) T_{n,1} \dots, T_{n,m_n}),$$
avec, d'après le lemme 88, $T_{1,1}, \dots, T_{n,m_n} \in \mathbb{G}^0$, et donc une unique façon d'indexer les sommets de $T_{1,1}, \dots, T_{n,m_n}$. Ainsi T appartient à $\mathbb{G}^0 \subseteq \mathbb{G}$ et il y a, dans ce cas, une seule façon de numéroter les sommets de T.
2. Si l'indice de la racine de T est plus grand que l'indice de son descendant direct le plus à gauche. Alors, pour tout $1 \leq i \leq n$,
$$\overbrace{B^+(\dots B^+}^{n-i \text{ fois}}(B^-(\bullet_1 T_1 \dots T_i) T_{i+1,1} \dots T_{i+1,m_{i+1}}) \dots) T_{n,1} \dots T_{n,m_n}) \in \mathbb{G}.$$
où, toujours avec le lemme 88, on doit avoir $T_1, \dots, T_i, T_{i+1,1}, \dots, T_{n,m_n} \in \mathbb{G}^0$ et il y a donc une unique façon d'indexer leurs sommets. Ainsi, cela fait n façons de numéroter les sommets de T dans ce cas.

Au final, il y a $1 + n$ façons d'indexer T pour obtenir un élément de $\mathbb{G}$. Le cas $k = 1$ est démontré car n est égal à la fertilité de la racine de T.

Si $k \geq 2$, $T = B_{NCK}(T_1 \dots T_n)$, avec $T_1 = B_{NCK}(T_{1,1} \dots T_{1,m_1}), \dots, T_n = B_{NCK}(T_{n,1} \dots T_{n,m_n})$ (si $T_i = \bullet$, $T_i = B_{NCK}(T_{i,1})$ avec $T_{i,1} = 1$). Il y a deux cas possibles pour indexer les sommets de T :

1. Si l'indice de la racine de T est plus petit que l'indice de son descendant direct le plus à gauche. Alors, pour que $T \in \mathbb{G}$,
$$T = B^+(\dots B^+(B^+(\bullet_1 T_{1,1} \dots T_{1,m_1}) \dots) T_{n,1} \dots T_{n,m_n}).$$
Avec le lemme 88, on doit avoir $T_{1,1}, \dots, T_{n,m_n} \in \mathbb{G}^0$, et il y a une unique façon d'indexer leurs sommets. On a ainsi une unique façon de numéroter T dans ce cas.
2. Si l'indice de la racine de T est plus grand que l'indice de son descendant direct le plus à gauche. Deux sous-cas sont possibles :

(a) Si l'indice de la racine de T_1 est plus petit que l'indice de son descendant direct le plus à gauche, c'est-à-dire si $T_1 = B^+(\dots B^+(\bullet_1 \dots)\dots)$, alors $T_1 \in \mathbb{G}^0$ (avec le lemme 88), et il y a une seule possibilité d'indexer T_1. Pour tout $1 \leq i \leq n$,

$$\overbrace{B^+(\dots B^+}^{n-i \text{ fois}}(B^-(T_1 \dots T_i)T_{i+1,1}\dots T_{i+1,m_{i+1}})\dots)T_{n,1}\dots T_{n,m_n}) \in \mathbb{G}.$$

D'après le lemme 88, $T_2, \dots, T_i, T_{i+1,1}, \dots, T_{n,m_n} \in \mathbb{G}^0$, donc il y a une unique façon de les indexer. On a ainsi n possibilités pour indexer T dans ce cas.

(b) Si l'indice de la racine de T_1 est plus grande que l'indice de son descendant direct le plus à gauche, c'est-à-dire si

$$T_1 = \overbrace{B^+(\dots B^+}^{m_1-i \text{ fois}}(B^-(T_{1,1}\dots T_{1,i})\dots)\dots),$$

pour un $1 \leq i \leq m_1$. Par hypothèse de récurrence, il y a $1 + \sum_{i=2}^{k-1} \prod_{L_T^l \twoheadrightarrow v \text{ et } 2 \leq h(v) \leq i} f(v)$ façons de numéroter $T_{1,1}$ pour obtenir un élément de $\mathbb{G}$. Toujours avec le lemme 88, il y a une unique façon de numéroter $T_{1,2}, \dots T_{1,m_1}$ car on doit nécessairement avoir après indexation $T_{1,2}, \dots, T_{1,m_1} \in \mathbb{G}^0$. Cela donne $m_1 \left(1 + \sum_{i=2}^{k-1} \prod_{L_T^l \twoheadrightarrow v \text{ et } 2 \leq h(v) \leq i} f(v)\right)$ possibilités pour indexer T_1. Comme l'indice de la racine de T doit être plus grande que l'indice de son descendant directe le plus à gauche, pour que $T \in \mathbb{G}$, on doit avoir

$$T = B^+(\dots B^+(B^-(T_1 \dots T_i)T_{i+1,1}\dots T_{i+1,m_{i+1}})\dots)T_{n,1}\dots T_{n,m_n}),$$

où $1 \leq i \leq n$. D'après le lemme 88, $T_2, \dots, T_i, T_{i+1,1}, \dots, T_{n,m_n} \in \mathbb{G}^0$ ont une unique indexation possible. Ainsi, dans ce cas, il y a $nm_1 \left(1 + \sum_{i=2}^{k-1} \prod_{L_T^l \twoheadrightarrow v \text{ et } 2 \leq h(v) \leq i} f(v)\right)$ possibilités d'indexer T.

En tout, cela fait bien

$$1 + \sum_{i=0}^{k-1} \prod_{L_T^l \twoheadrightarrow v \text{ et } h(v) \leq i} f(v)$$

possibilités pour indexer les sommets de T. Le principe de récurrence permet de conclure. □

Dans le cas des échelles, il existe un résultat plus précis :

Corollaire 92 *Pour tout $n \geq 1$ et pour tout $i \in \{1, \dots, n\}$, il existe une unique échelle (de degré n) dans chaque sous-ensemble $\mathbb{G}^{(\overbrace{+,\dots,+}^{i \text{ fois}},\overbrace{-,\dots,-}^{n-i \text{ fois}})}$.*

Preuve. Raisonnons par récurrence sur n. Le résultat étant trivial pour $n = 1, 2$, supposons $n \geq 3$. D'après l'hypothèse de récurrence, pour tout $i \in \{1, \dots, n-1\}$, il existe une unique échelle T_i de degré $n-1$ dans chaque sous-ensemble $\mathbb{G}^{(\overbrace{+,\dots,+}^{i \text{ fois}},\overbrace{-,\dots,-}^{n-1-i \text{ fois}})}$. En réalisant un B^-, on construit $n-1$ échelles $E_i = B^-(T_i)$ de degré n, et $\forall\, i \in \{1, \dots, n-1\}$, $E_i \in \mathbb{G}^{(\overbrace{+,\dots,+}^{i \text{ fois}},\overbrace{-,\dots,-}^{n-i \text{ fois}})}$. De plus, toujours par hypothèse de récurrence, il existe une unique échelle T de degré $n-2$ appartenant à $\mathbb{G}^0$. Alors, $\bullet_1 T \in \mathbb{G}^0$, en utilisant le lemme 89, et donc $E_n = B^+(\bullet_1 T) \in \mathbb{G}^0$ et c'est par construction une échelle.

Donc $\forall\, i \in \{1, \dots, n\}$, $E_i \in \mathbb{G}^{(\overbrace{+,\dots,+}^{i \text{ fois}},\overbrace{-,\dots,-}^{n-i \text{ fois}})}$. D'après le lemme 87, les échelles E_i sont toutes distinctes. Comme il existe exactement n échelles de degré n dans $\mathbb{G}$ (d'après la proposition 91), il y a donc une unique échelle dans chaque sous-ensemble $\mathbb{G}^{(\overbrace{+,\dots,+}^{i \text{ fois}},\overbrace{-,\dots,-}^{n-i \text{ fois}})}$, pour $i \in \{1, \dots, n\}$. Le résultat est ainsi démontré au rang n et on conclut par le principe de récurrence. □

4.1.2 L'algèbre de Hopf $\mathbf{B}^\infty$

Notons $\mathbf{B}^\infty = \mathbb{K}(\mathbb{G} \cup \{1\})$ l'algèbre engendrée par $\mathbb{G} \cup \{1\}$. D'après le lemme 88, $\mathbf{B}^\infty$ est engendrée librement par les arbres appartenant à $\mathbb{G}$.

Nous avons le résultat remarquable suivant :

Proposition 93 *L'algèbre $\mathbf{B}^\infty$ est une algèbre de Hopf.*

Preuve. Il suffit de montrer que, en réalisant une coupe simple d'un arbre appartenant à $\mathbb{G}$, la branche et le tronc appartiennent respectivement à $\mathbb{G}$ et $\mathbb{G} \cup \{1\}$. En effet, si ce résultat est démontré, on aura alors le résultat pour une coupe admissible quelconque puisque $\mathbf{B}^\infty$ est engendrée par $\mathbb{G} \cup \{1\}$ comme algèbre. Travaillons par récurrence sur le degré des arbres. Le résultat est trivial pour $n = 2, 3$. Au rang $n \geq 4$, considérons un arbre $T \in \mathbb{G}$ de degré n, et $\boldsymbol{v} \models V(T)$ une coupe simple. Il y a deux cas possibles :

1. Si l'arbre est de la forme $T = B^-(T_1 \dots T_m) \in \mathbb{G}^{(\dots,-)}$. Par construction, la forêt $T_1 \dots T_m \in \mathbb{G}$ et, avec le lemme 88, $T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$. Si $\boldsymbol{v}$ est la coupe totale, le résultat est trivial. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (on inclut ici le cas de la coupe totale qui correspond à couper l'arête entre la racine de T et celle de T_i). Par récurrence, $Lea_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{G}$, car $T_i \in \mathbb{G}$. De même, par récurrence, $Roo_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{G} \cup \{1\}$. Alors la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m \in \mathbb{G} \cup \{1\}$ car :
   (a) si $i = 1$, $Roo_{\boldsymbol{v}}(T_1) \in \mathbb{G} \cup \{1\}$ et comme $T_2, \dots, T_m \in \mathbb{G}^0$, $Roo_{\boldsymbol{v}}(T_1) T_2 \dots T_m \in \mathbb{G} \cup \{1\}$ en utilisant le lemme 88.
   (b) si $i \geq 2$, $T_i \in \mathbb{G}^0$ donc, d'après le lemme 90, $Roo_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{G}^0 \cup \{1\}$. Ainsi, toujours avec le lemme 88, la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m$ appartient à $\mathbb{G}$.

   Donc $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i) \in \mathbb{G}$ et $Roo_{\boldsymbol{v}}(T) = B^-(T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m) \in \mathbb{G} \cup \{1\}$.
2. Si l'arbre est de la forme $T = B^+(T_1, \dots, T_m) \in \mathbb{G}^{(\dots,+)}$. Par construction, la forêt $T_1 \dots T_m \in \mathbb{G}$, donc $T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$. Si $\boldsymbol{v}$ est la coupe simple correspondant à couper l'arête joignant la racine de T (qui est la racine de T_1) et le sommet indexé par n joignant les racines communes de $T_2, \dots, T_m$, alors $Roo_{\boldsymbol{v}}(T) = T_1 \in \mathbb{G}$ et $Lea_{\boldsymbol{v}}(T) = B^-(T_2 \dots T_m) \in \mathbb{G}$. Le résultat est donc vérifié dans ce cas. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (si $i \geq 2$, le cas de la coupe totale correspond à couper l'arête entre le sommet de T indexé par n et la racine de T_i). Il y a alors deux cas à distinguer :
   (a) Si $i = 1$, c'est-à-dire si $\boldsymbol{v} \models V(T_1)$. Si $\boldsymbol{v}$ est totale, alors $Lea_{\boldsymbol{v}}(T) = T$ et $Roo_{\boldsymbol{v}}(T) = 1$ et le résultat est trivial. Sinon, par récurrence, $Lea_{\boldsymbol{v}}(T_1) \in \mathbb{G}$, et $Roo_{\boldsymbol{v}}(T_1)$ étant un arbre non vide $Roo_{\boldsymbol{v}}(T_1) \in \mathbb{G}$. Ainsi, avec le lemme 88, $Roo_{\boldsymbol{v}}(T_1) T_2 \dots T_m \in \mathbb{G}$. D'où $Roo_{\boldsymbol{v}}(T) = B^+(Roo_{\boldsymbol{v}}(T_1) T_2 \dots T_m)$ et $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_1)$ appartiennent à $\mathbb{G}$.
   (b) Si $i \geq 2$, c'est-à-dire si $\boldsymbol{v} \models V(T_i)$, toujours par récurrence, $Lea_{\boldsymbol{v}}(T_i) \in \mathbb{G}$ et avec le lemme 90, $Roo_{\boldsymbol{v}}(T_i) \in \mathbb{G}^0 \cup \{1\}$. Donc la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m \in \mathbb{G}$ et on a ainsi $Roo_{\boldsymbol{v}}(T) = B^+(T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m)$ et $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i)$ qui sont des éléments de $\mathbb{G}$.

   Dans tous les cas, $Roo_{\boldsymbol{v}}(T) \in \mathbb{G} \cup \{1\}$, $Lea_{\boldsymbol{v}}(T) \in \mathbb{G}$.

Par récurrence, le résultat est démontré. □

Proposition 94 *La série formelle de l'algèbre de Hopf $\mathbf{B}^\infty$ est donnée par la formule :*

$$F_{\mathbf{B}^\infty}(x) = \frac{1}{2\sqrt{1-4x}} + \frac{1}{2}.$$

Preuve. Pour calculer la série formelle de l'algèbre $\mathbf{B}^\infty$, nous introduisons quelques notations. Posons $f_{i,j}^{\mathbf{B}^\infty}$ le nombre de forêts de longueur i et de degré j, et $f_j^{\mathbf{B}^\infty}$ le nombre de forêts de degré j. En particulier, $f_j^{\mathbf{B}^\infty} = \sum_{1 \leq i \leq j} f_{i,j}^{\mathbf{B}^\infty}$. Par construction de $\mathbf{B}^\infty$, on a les relations suivantes :

$$\begin{aligned}
f_{1,1}^{\mathbf{B}^\infty} &= 1 \\
f_{k,1}^{\mathbf{B}^\infty} &= 0 \text{ si } k \geq 2 \\
f_{1,n}^{\mathbf{B}^\infty} &= 2 f_{n-1}^{\mathbf{B}^\infty} \\
f_{k,n}^{\mathbf{B}^\infty} &= f_{k-1,n-1}^{\mathbf{B}^\infty} + \dots + f_{n-1,n-1}^{\mathbf{B}^\infty} \text{ si } k \geq 2, n \geq 2.
\end{aligned}$$

Posons $F(x) = \sum_{i\geq 1} f_i^{\mathbf{B}^\infty} x^i$ et, pour $k \geq 1$, $F_k(x) = \sum_{i\geq 1} f_{k,i}^{\mathbf{B}^\infty} x^i$. Comme $f_{k,i}^{\mathbf{B}^\infty} = 0$ si $i < k$, $F_k(x) = \sum_{i\geq k} f_{k,i}^{\mathbf{B}^\infty} x^i$. Alors,

$$\begin{aligned}
F(x) &= \sum_{k\geq 1} F_k(x)\\
F_1(x) &= 2xF(x) + x\\
F_k(x) &= x\left(\sum_{i\geq k-1} F_i(x)\right) = x\Big(F(x) - F_1(x) - \ldots - F_{k-2}(x)\Big) \text{ si } k \geq 2.
\end{aligned}$$

Par différence, on obtient pour tout $l \geq 1$:

$$F_{l+2} - F_{l+1} + xF_l = 0.$$

Il existe donc $A(x), B(x) \in \mathbb{C}[[x, x^{-1}]]$ tels que $\forall l \geq 1$,

$$F_l(x) = A(x)\left(\frac{1-\sqrt{1-4x}}{2}\right)^l + B(x)\left(\frac{1+\sqrt{1-4x}}{2}\right)^l,$$

d'où

$$F(x) = A(x)\frac{1-2x-\sqrt{1-4x}}{2} + B(x)\frac{1-2x+\sqrt{1-4x}}{2}. \tag{4.2}$$

En faisant $l = 1, 2$, nous avons les deux relations suivantes :

$$\left\{\begin{aligned}
A(x) &= \frac{1+\sqrt{1-4x}}{2}F(x) + \frac{1}{2} + \frac{1-2x}{2\sqrt{1-4x}}\\
B(x) &= \frac{1-\sqrt{1-4x}}{2}F(x) + \frac{1}{2} + \frac{1-2x}{2\sqrt{1-4x}}
\end{aligned}\right. \tag{4.3}$$

Montrons que $B(x) = 0$. Si $B(x) \neq 0$, $B(x) = a_k x^k + \ldots$, $a_k \neq 0$. De plus,

$$F_l(x) = A(x)\Bigg(\underbrace{\frac{1-\sqrt{1-4x}}{2}}_{=x+\ldots}\Bigg)^l + B(x)\Bigg(\underbrace{\frac{1+\sqrt{1-4x}}{2}}_{=1+\ldots}\Bigg)^l,$$

donc, si $l > k$,

$$\begin{aligned}
A(x)\left(\frac{1-\sqrt{1-4x}}{2}\right)^l &= \mathcal{O}(x^l)\\
B(x)\left(\frac{1+\sqrt{1-4x}}{2}\right)^l &= a_k x^k + \ldots
\end{aligned}$$

D'où $F_l(x) = a_k x^k + \ldots$. Or $F_l(x) = \sum_{i\geq l} f_{l,i}^{\mathbf{B}^\infty} x^i$, donc $F_l(x) = \mathcal{O}(x^l)$, et $a_k = 0$.

Ainsi $B(x) = 0$, et avec (4.2) et (4.3),

$$\begin{aligned}
F(x) &= \frac{1-4x-\sqrt{1-4x}}{2(4x-1)} = \frac{1}{2\sqrt{1-4x}} - \frac{1}{2}\\
A(x) &= \frac{1}{2} + \frac{1}{2\sqrt{1-4x}}
\end{aligned}$$

Au passage, on obtient la formule pour les F_k :

$$F_k(x) = \frac{x}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2}\right)^{k-1}.$$

Finalement la série formelle de $\mathbf{B}^\infty$ est donnée par :

$$F_{\mathbf{B}^\infty}(x) = \frac{1}{2\sqrt{1-4x}} + \frac{1}{2}.$$

□

Ainsi, pour tout $k \geq 1$, $f_{1,k}^{\mathbf{B}^\infty} = \dfrac{(2k-2)!}{(k-1)!(k-1)!}$ et $f_k^{\mathbf{B}^\infty} = \dfrac{(2k)!}{2(k!)^2}$. Voici quelques valeurs numériques :

k	1	2	3	4	5	6	7	8
$f_{1,k}^{\mathbf{B}^\infty}$	1	2	6	20	70	252	924	3432
$f_k^{\mathbf{B}^\infty}$	1	3	10	35	126	462	1716	6435

Ce sont les séquences A000984 et A001700 de [Slo].

4.1.3 La sous-algèbre de Hopf $\mathbf{B}^1$

Muni de la concaténation, l'ensemble $\mathbb{G}^0 \cup \{1\}$, constitué de l'arbre vide et de tous les arbres construits uniquement avec des B^+, est un monoïde. Notons $\mathbf{B}^0$ l'algèbre unitaire engendrée par ce monoïde. Si T est un arbre appartenant à $\mathbb{G}^0$, il existe certaines coupes $\boldsymbol{v} \models V(T)$ telles que $Lea_{\boldsymbol{v}}(T) \notin \mathbb{G}^0 \cup \{1\}$ (voir la remarque qui suit le lemme 90). $\mathbf{B}^0$ n'est donc pas une cogèbre. Par contre, d'après le lemme 90, $\mathbf{B}^0$ est un comodule à droite de l'algèbre de Hopf $\mathbf{B}^\infty$.

Remarquons que $\mathbf{B}^0$ est isomorphe en tant qu'algèbre à l'algèbre (de Hopf) des arbres enracinés plans $\mathbf{H}_{NCK}$. En effet, on a vu au paragraphe qui précède la proposition 91 que les arbres de $\mathbb{G}^0$ sont en bijection avec ceux de $\mathbf{H}_{NCK}$. Cette bijection s'étend en un isomorphisme d'algèbres graduées de $\mathbf{H}_{NCK}$ dans $\mathbf{B}^0$. En particulier, ces deux algèbres ont la même série formelle :

$$F_{\mathbf{B}^0}(x) = \frac{1-\sqrt{1-4x}}{2x} = F_{\mathbf{H}_{NCK}}(x).$$

Considérons maintenant l'ensemble

$$\mathbb{G}^1 = \bigcup_{n \geq 1, \varepsilon_n \in \{+,-\}} \mathbb{G}^{(\overbrace{+,\ldots,+}^{n-1 \text{ fois}},\varepsilon_n)}.$$

Notons $\mathbf{B}^1$ l'algèbre $\mathbb{K}\left(\mathbb{G}^1 \cup \{1\}\right)$. Alors :

Proposition 95 *L'algèbre $\mathbf{B}^1$ est une algèbre de Hopf.*

Preuve. Il suffit de montrer que, étant donné un arbre appartenant à $\mathbb{G}^1$, la branche et le tronc de cet arbre, après avoir réalisé une coupe simple, appartiennent respectivement à $\mathbb{G}^1$ et $\mathbb{G}^1 \cup \{1\}$. On travaille par récurrence sur le degré n des arbres de $\mathbf{B}^1$. Ceci est évident pour $n \leq 3$, en le vérifiant rapidement à la main.

Au rang $n \geq 4$. Considérons un arbre $T \in \mathbb{G}^1$ de degré n et $\boldsymbol{v} \models V(T)$ une coupe simple. Il y a deux cas :

1. Si T est de la forme $B^-(T_1 \ldots T_m) \in \mathbb{G}^{(+,\ldots,+,-)}$, avec $T_1, \ldots, T_m \in \mathbb{G}^0$. Si $\boldsymbol{v}$ est la coupe totale, le résultat est trivial. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \ldots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (on inclut le cas de la coupe totale qui correspond à couper l'arête entre la racine de T et celle de T_i). Comme $T_i \in \mathbb{G}^0$, par récurrence, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i) \in \mathbb{G}^1$. D'autre part, avec le lemme 90, T_i appartenant à $\mathbb{G}^0$, $Roo_{\boldsymbol{v}}(T_i) \in \mathbb{G}^0 \cup \{1\}$. Donc la forêt $T_1 \ldots Roo_{\boldsymbol{v}}(T_i) \ldots T_m$ appartient à $\mathbb{G}^0 \cup \{1\}$, et $Roo_{\boldsymbol{v}}(T) = B^-(T_1 \ldots Roo_{\boldsymbol{v}}(T_i) \ldots T_m) \in \mathbb{G}^1$. Le résultat est ainsi démontré si T est de la forme $B^-(T_1 \ldots T_m) \in \mathbb{G}^{(+,\ldots,+,-)}$.
2. Si T est de la forme $B^+(T_1 \ldots T_m) \in \mathbb{G}^{(+,\ldots,+)}$, avec $T_1, \ldots, T_m \in \mathbb{G}^0$. Si $\boldsymbol{v}$ est la coupe simple correspondant à couper l'arête joignant la racine de T (qui est la racine de T_1) et le sommet indexé par n joignant les racines communes de $T_2, \ldots, T_m$, alors $Roo_{\boldsymbol{v}}(T) = T_1 \in \mathbb{G}^0$ et $Lea_{\boldsymbol{v}}(T) = B^-(T_2 \ldots T_m) \in \mathbb{G}^1$. Le résultat est donc démontré dans ce cas. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \ldots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (on inclut le cas de la coupe totale qui, si $i \geq 2$, correspond à couper l'arête entre le sommet de T indexé par n et la racine de T_i). Il y a alors deux cas :

(a) Si $i = 1$, c'est-à-dire si $\boldsymbol{v} \models V(T_1)$. Si $\boldsymbol{v}$ est la coupe totale, alors $Lea_{\boldsymbol{v}}(T) = T$ et $Roo_{\boldsymbol{v}}(T) = 1$ et le résultat est établi. Sinon, par récurrence, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_1) \in \mathbb{G}^1$. Avec le lemme 90, comme $Roo_{\boldsymbol{v}}(T_1)$ est différent de l'arbre vide, $Roo_{\boldsymbol{v}}(T_1) \in \mathbb{G}^0$, donc $Roo_{\boldsymbol{v}}(T_1)T_2 \dots T_m \in \mathbb{G}^0$. Et ainsi $Roo_{\boldsymbol{v}}(T) = B^+(Roo_{\boldsymbol{v}}(T_1)T_2 \dots T_m) \in \mathbb{G}^0 \subseteq \mathbb{G}^1 \cup \{1\}$.

(b) Si $i \geq 2$. Par récurrence, comme $T_i \in \mathbb{G}^0$, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i) \in \mathbb{G}^1$. D'autre part, avec le lemme 90, $Roo_{\boldsymbol{v}}(T_i) \in \mathbb{G}^0 \cup \{1\}$. Donc la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m$ appartient à $\mathbb{G}^0$, et $Roo_{\boldsymbol{v}}(T) = B^+(T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m) \in \mathbb{G}^0 \subseteq \mathbb{G}^1 \cup \{1\}$.

Ceci démontre la stabilité de $\mathbf{B}^1$ par coupe simple, et donc par coupe admissible, car $\mathbf{B}^1$ est stable pour le produit. Ainsi $\mathbf{B}^1$ est bien une algèbre de Hopf. □

Remarque. On peut voir en reprenant cette démonstration que l'algèbre $\mathbf{B}^0$ est même un comodule à droite de l'algèbre de Hopf $\mathbf{B}^1$.

4.1.4 Généralisation de l'algèbre de Hopf $\mathbf{B}^1$

En utilisant le même modèle que pour la construction de l'algèbre de Hopf $\mathbf{B}^1$, nous allons à présent construire une infinité d'algèbres de Hopf $\mathbf{B}^i$, pour $i \geq 2$, telles qu'on ait les relations d'inclusions suivantes :

$$\mathbf{B}^1 \subseteq \mathbf{B}^2 \subseteq \dots \subseteq \mathbf{B}^i \subseteq \mathbf{B}^{i+1} \subseteq \dots \subseteq \mathbf{B}^\infty.$$

Pour cela, posons, quelque soit $i \geq 2$,

$$\mathbb{G}^i = \bigcup_{n \geq 1, (\varepsilon_{n-i+1}, \dots, \varepsilon_n) \in \{+,-\}^i} \mathbb{G}^{(\overbrace{+, \dots, +}^{n-i \text{ fois}}, \varepsilon_{n-i+1}, \dots, \varepsilon_n)}.$$

Notons $\mathbf{B}^i = \mathbb{K}\left(\mathbb{G}^i \cup \{1\}\right)$. Remarquons que $\mathbb{G}^i \subseteq \mathbb{G}^{i+1}$, $\forall i \geq 1$. Comme annoncé, on a alors le résultat suivant :

Proposition 96 *Pour tout $i \geq 1$, l'algèbre $\mathbf{B}^i$ est une algèbre de Hopf.*

Preuve. Travaillons par récurrence sur $i \geq 1$. Le cas $i = 1$ ayant déjà été démontré, supposons $i \geq 2$.

Il faut montrer que $\mathbf{B}^i$ est stable par coupe simple. Pour cela, faisons une récurrence sur le degré n des arbres de $\mathbb{G}^i$. Ceci est clair pour $n \leq 3$ et découle directement du fait que $\mathbf{B}^\infty$ est une algèbre de Hopf.

Au rang $n \geq 4$. Considérons un arbre $T \in \mathbb{G}^i$ de degré n et $\boldsymbol{v} \models V(T)$ une coupe simple. Ici encore, il faut distinguer deux cas :

1. Si T est de la forme $B^-(T_1 \dots T_m) \in \mathbb{G}^{(\dots,-)}$. D'après le lemme 88, $T_1 \in \mathbb{G}$ et $T_2, \dots, T_m \in \mathbb{G}^0$. Comme $T \in \mathbb{G}^i$, la forêt $T_1 \dots T_m$ appartient à $\mathbb{G}^{i-1}$. En posant $k = |T_2|_v + \dots + |T_m|_v$, $T_1 \in \mathbb{G}^{i-1-k}$. Supposons $\boldsymbol{v}$ différent de la coupe totale, le résultat étant trivial dans ce cas. Comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $j \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_j)$ (on inclut le cas de la coupe totale qui correspond à couper l'arête entre la racine de T et celle de T_j). Comme $T_j \in \mathbb{G}^i$, par récurrence, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_j) \in \mathbb{G}^i$. Pour $Roo_{\boldsymbol{v}}(T)$, on doit alors différencier deux sous-cas :
   (a) Si $j = 1$. T_1 appartient à $\mathbb{G}^{i-1-k}$. Par récurrence, $Roo_{\boldsymbol{v}}(T_1)$ est un arbre appartenant à $\mathbb{G}^{i-1-k} \cup \{1\}$. D'après le lemme 88, la forêt $Roo_{\boldsymbol{v}}(T_1)T_2 \dots T_m$ appartient à $\mathbb{G}^{i-1} \cup \{1\}$. Donc $Roo_{\boldsymbol{v}}(T) = B^-(Roo_{\boldsymbol{v}}(T_1)T_2 \dots T_m) \in \mathbb{G}^i$.
   (b) Si $j \geq 2$. Par le lemme 90, $Roo_{\boldsymbol{v}}(T_j) \in \mathbb{G}^0 \cup \{1\}$. Notons l la somme $|T_2|_v + \dots + |Roo_{\boldsymbol{v}}(T_j)|_v + \dots + |T_m|_v$. Alors $l \leq k$ et la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_j) \dots T_m$ appartient à $\mathbb{G}^{i-1+l-k} \subseteq \mathbb{G}^{i-1}$. Et ainsi $Roo_{\boldsymbol{v}}(T) = B^-(T_1 \dots Roo_{\boldsymbol{v}}(T_j) \dots T_m) \in \mathbb{G}^i$.

   Ceci termine la démonstration dans le premier cas.
2. Si T est de la forme $B^+(T_1 \dots T_m) \in \mathbb{G}^{(\dots,+)}$. Avec le même raisonnement que dans le premier cas, en notant $k = |T_2|_v + \dots + |T_m|_v$, $T_1 \in \mathbb{G}^{i-1-k}$ et $T_2, \dots, T_m \in \mathbb{G}^0$. Si $\boldsymbol{v}$ est la coupe simple correspondant à couper l'arête joignant la racine de T (qui est la racine de T_1) et le sommet indexé par n joignant les racines communes de $T_2, \dots, T_m$, alors $Roo_{\boldsymbol{v}}(T) = T_1 \in \mathbb{G}^{i-1-k} \subseteq \mathbb{G}^i$ et $Lea_{\boldsymbol{v}}(T) = B^-(T_2 \dots T_m) \in \mathbb{G}^1 \subseteq \mathbb{G}^i$. Le résultat est donc démontré dans ce cas. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $j \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_j)$ (on inclut le cas de la coupe totale qui, si $j \geq 2$, correspond à couper l'arête entre le sommet de T indexé par n et la racine de T_j). Il y a alors deux sous-cas :

(a) Si $j=1$. Dans le cas où $\boldsymbol{v}$ est la coupe totale, $Lea_{\boldsymbol{v}}(T)=T$ et $Roo_{\boldsymbol{v}}(T)=1$ et le résultat est trivial. Sinon, par récurrence, comme T_1 appartient à $\mathbb{G}^{i-1-k}$, $Lea_{\boldsymbol{v}}(T)=Lea_{\boldsymbol{v}}(T_1)\in\mathbb{G}^{i-1-k}\subseteq\mathbb{G}^i$ et $Roo_{\boldsymbol{v}}(T_1)$ est un arbre appartenant à $\mathbb{G}^{i-1-k}$. Avec le lemme 88, $Roo_{\boldsymbol{v}}(T_1)T_2\ldots T_m\in\mathbb{G}^{i-1}$. Donc $Roo_{\boldsymbol{v}}(T)=B^+(Roo_{\boldsymbol{v}}(T_1)T_2\ldots T_m)\in\mathbb{G}^i$.

(b) Si $j\geq 2$. Comme $T_j\in\mathbb{G}^0$, par la proposition 95, $Lea_{\boldsymbol{v}}(T)=Lea_{\boldsymbol{v}}(T_j)\in\mathbb{G}^1\subseteq\mathbb{G}^i$. Par le lemme 90, $Roo_{\boldsymbol{v}}(T_j)\in\mathbb{G}^0\cup\{1\}$. Si on note l la somme $|T_2|_v+\ldots+|Roo_{\boldsymbol{v}}(T_j)|_v+\ldots+|T_m|_v$, alors $l\leq k$ et la forêt $T_1\ldots Roo_{\boldsymbol{v}}(T_j)\ldots T_m$ appartient à $\mathbb{G}^{i-1+l-k}\subseteq\mathbb{G}^{i-1}$. Ainsi $Roo_{\boldsymbol{v}}(T)=B^-(T_1\ldots Roo_{\boldsymbol{v}}(T_j)\ldots T_m)\in\mathbb{G}^i$.

Le deuxième cas est donc démontré.

Nous avons ainsi obtenu, par le principe de récurrence, le résultat annoncé. □

Remarque. En reprenant la démonstration, on a, pour tout $i\geq 2$, $\tilde{\Delta}(\mathbf{B}^i)\subseteq\mathbf{B}^i\otimes\mathbf{B}^{i-1}$.

Il est alors possible de calculer le nombre d'arbres de degré k de $\mathbf{B}^i$:

Proposition 97 *Soit $k\geq 1$. Notons, pour tout $i\geq 1$, $f_{1,k}^{\mathbf{B}^i}$ le nombre d'arbres de $\mathbf{B}^i$ de degré k. Rappelons que $f_{1,k}^{\mathbf{B}^\infty}$ désigne le nombre d'arbres de $\mathbf{B}^\infty$ de degré k, et $f_k^{\mathbf{B}^0}$ le nombre de forêts de $\mathbf{B}^0$ de degré k. Alors, pour tout $k\geq 1$, $i\geq 1$,*

$$f_{1,k}^{\mathbf{B}^i}=\begin{cases} f_{1,k}^{\mathbf{B}^\infty} & \text{si } k\leq i+1,\\ f_{1,k}^{\mathbf{B}^\infty}-\displaystyle\sum_{1\leq j\leq k-i-1} f_j^{\mathbf{B}^0}f_{1,k-j}^{\mathbf{B}^\infty} & \text{si } k\geq i+2.\end{cases}$$

Preuve. Fixons $i\in\mathbb{N}^*$. La formule est évidente pour $k\leq i+1$ (en utilisant l'identification $\mathbb{G}^{(+,\underline{\varepsilon})}=\mathbb{G}^{(-,\underline{\varepsilon})}$ lorsque $k=i+1$). Supposons $k\geq i+1$. Notons A^k l'ensemble

$$\left\{\underline{\varepsilon}\in\{+,-\}^k\mid \varepsilon_1=\ldots=\varepsilon_{k-i}=+\right\}\subseteq\{+,-\}^k.$$

Le complémentaire dans $\{+,-\}^k$ de A^k est $\displaystyle\bigcup_{0\leq j\leq k-i-1}A_j^k$ (l'union étant disjointe) où

$$A_j^k=\left\{\underline{\varepsilon}\in\{+,-\}^k\mid \varepsilon_1=\ldots=\varepsilon_j=+\text{ et }\varepsilon_{j+1}=-\right\}\subseteq\{+,-\}^k.$$

D'après le lemme 87, pour tout $0\leq j\leq k-i-1$, les ensembles $\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in A_j^k\right\}$ sont disjoints. De plus

$$\begin{aligned}\mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in\{+,-\}^k\right\}\right)&=f_{1,k+1}^{\mathbf{B}^\infty},\\ \mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in A_j^k\right\}\right)&=f_{j+1}^{\mathbf{B}^0}f_{1,k-j}^{\mathbf{B}^\infty}.\end{aligned}$$

Alors

$$\begin{aligned}f_{1,k+1}^{\mathbf{B}^i}&=\mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in A^k\right\}\right)\\ &=\mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in\{+,-\}^k\right\}\right)\\ &\qquad-\mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in\bigcup_{0\leq j\leq k-i-1}A_j^k\right\}\right)\\ &=f_{1,k+1}^{\mathbf{B}^\infty}-\mathrm{card}\left(\bigcup_{0\leq j\leq k-i-1}\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in A_j^k\right\}\right)\\ &=f_{1,k+1}^{\mathbf{B}^\infty}-\sum_{0\leq j\leq k-i-1}\mathrm{card}\left(\left\{\text{arbre de }\mathbb{G}^{(+,\underline{\varepsilon})}\mid\underline{\varepsilon}\in A_j^k\right\}\right)\\ &=f_{1,k+1}^{\mathbf{B}^\infty}-\sum_{0\leq j\leq k-i-1}f_{j+1}^{\mathbf{B}^0}f_{1,k-j}^{\mathbf{B}^\infty}.\end{aligned}$$

□

Voici quelques valeurs numériques :

1. Pour les arbres :

k	1	2	3	4	5	6	7	8
$f_{1,k}^{\mathbf{B}^0}$	1	1	2	5	14	42	132	429
$f_{1,k}^{\mathbf{B}^1}$	1	2	4	10	28	84	264	858
$f_{1,k}^{\mathbf{B}^2}$	1	2	6	14	38	112	348	1122
$f_{1,k}^{\mathbf{B}^3}$	1	2	6	20	50	142	432	1374
$f_{1,k}^{\mathbf{B}^4}$	1	2	6	20	70	182	532	1654
$f_{1,k}^{\mathbf{B}^5}$	1	2	6	20	70	252	672	2004
$f_{1,k}^{\mathbf{B}^6}$	1	2	6	20	70	252	924	2508

2. Pour les forêts :

k	1	2	3	4	5	6	7	8
$f_{k}^{\mathbf{B}^0}$	1	2	5	14	42	132	429	1430
$f_{k}^{\mathbf{B}^1}$	1	3	9	29	97	333	1165	4135
$f_{k}^{\mathbf{B}^2}$	1	3	11	37	129	461	1669	6107
$f_{k}^{\mathbf{B}^3}$	1	3	11	43	153	557	2065	7739
$f_{k}^{\mathbf{B}^4}$	1	3	11	43	173	637	2385	9059
$f_{k}^{\mathbf{B}^5}$	1	3	11	43	173	707	2665	10179
$f_{k}^{\mathbf{B}^6}$	1	3	11	43	173	707	2917	11187

4.2 L'algèbre de Hopf B

4.2.1 Construction de B et premières propriétés

A partir d'une construction similaire à celle décrite dans la partie 4.1.1, nous allons définir une nouvelle algèbre notée $\mathbf{B}$ contenant l'algèbre de Hopf $\mathbf{B}^\infty$. Pour cela, on construit inductivement un ensemble d'arbres $\mathbb{T}^{(n,\varepsilon)}$ de degré $n \geq 1$, pour $\varepsilon \in \{+,-\}$.

Si $\underline{n=1}$, $\mathbb{T}^{(1,\varepsilon)}$, pour ε quelconque, est l'ensemble réduit à un seul élément, l'unique arbre de degré 1.

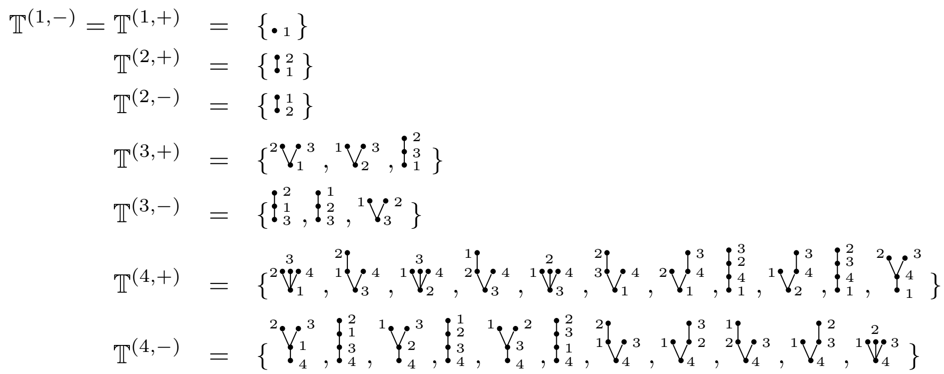
Si $\underline{n \geq 2}$, supposons les ensembles $\mathbb{T}^{(k,\varepsilon)}$ construits, pour tout $1 \leq k \leq n-1$.

1. Si $\varepsilon = -$, les arbres T de $\mathbb{T}^{(n,-)}$ sont construits comme suit. On prend une forêt $F = T_1 \dots T_m$ de degré $n-1$ construite à partir de $m \geq 1$ arbres appartenant à l'ensemble $\bigcup_{1 \leq k \leq n-1, \varepsilon \in \{+,-\}} \mathbb{T}^{(k,\varepsilon)}$ et on considère $T_1, \dots, T_m$ comme la suite des sous-arbres d'un arbre enraciné ayant pour racine le sommet indexé par n. Cela donne ainsi naissance à un nouvel arbre ordonné $T = B^-(T_1 \dots T_m)$ de degré n.

2. Si $\varepsilon = +$, on construit alors les arbres T de $\mathbb{T}^{(n,+)}$ par la transformation suivante. On prend une forêt $F = T_1 \dots T_m$ de degré $n-1$ avec $T_1, \dots, T_m \in \bigcup_{1 \leq k \leq n-1, \varepsilon \in \{+,-\}} \mathbb{T}^{(k,\varepsilon)}$ et $m \geq 1$. On construit alors T en greffant le sommet indexé par n comme le fils le plus à droite de la racine de T_1 et en considerant $T_2, \dots, T_m$ comme la suite des sous-arbres issus du sommet indexé par n. Cela donne un nouvel arbre ordonné $T = B^+(T_1 \dots T_m)$ de degré n.

Considérons un arbre T construit à partir des instructions précédentes, de degré n. Soit le sommet indexé par n est la racine et $T \in \mathbb{T}^{(n,-)}$, soit le sommet indexé par n est le fils le plus à droite de la racine et $T \in \mathbb{T}^{(n,+)}$. De plus, deux arbres de $\mathbb{T}^{(n,-)}$ (resp. $\mathbb{T}^{(n,+)}$) sont égaux si et seulement si les forêts à partir desquelles ils sont construits sont égales. Ainsi, les arbres construits avec les instructions précédentes sont tous distincts. En particulier, $\mathrm{card}\left(\mathbb{T}^{(n,+)}\right) = \mathrm{card}\left(\mathbb{T}^{(n,-)}\right)$, $\forall\, n \geq 1$.

On pose alors $\mathbb{T}$ l'ensemble $\bigcup_{n \geq 1, \varepsilon \in \{+,-\}} \mathbb{T}^{(n,\varepsilon)}$, l'union étant disjointe (on identifie $\mathbb{T}^{(1,-)}$ et $\mathbb{T}^{(1,+)}$). Voici une illustration pour $n = 1, 2, 3$ et 4 :



Soit T un arbre enraciné plan non vide $\in \mathbf{H}_{NCK}$. Ici aussi il est possible de dénombrer le nombre de façons d'indexer les sommets de T pour obtenir un élément de $\mathbb{T}$:

Proposition 98 *Soit T un arbre plan. On définit pour chaque sommet v de T, un entier a_v par récurrence sur la hauteur :*

1. *Si v est une feuille, $a_v = 1$.*
2. *Sinon,*

$$a_v = \prod_{v'\to v} a_{v'} + \sum_{v'\to v} \left(\prod_{v''\to v \text{ tq } v''<v'} a_{v''} \right) \left(\prod_{v''\to w\to v \text{ tq } v'\leq w} a_{v''} \right),$$

l'ordre sur les sommets de T étant celui défini dans l'introduction.

Alors, il y a a_{R_T} façons d'indexer les sommets de T pour obtenir un élément de $\mathbb{T}$.

Preuve. Pour éviter d'alourdir les notations, on notera indifféremment un arbre appartenant à $\mathbf{H}_{NCK}$ et l'arbre (ou les arbres) obtenu après indexation des sommets appartenant à $\mathbb{T}$.

Par récurrence sur le degré k de $T \in \mathbb{T}$. Si $k = 1$, c'est trivial. Supposons $k \geq 2$ et le résultat vérifié au rang inférieur. On utilise les mêmes notations que pour la démonstration de la proposition 91 : $T = B(T_1 \dots T_n)$, avec $T_1 = B(T_{1,1} \dots T_{1,m_1}), \dots, T_n = B(T_{n,1} \dots T_{n,m_n})$ (si $T_i = \bullet$, $T_i = B(T_{i,1})$ avec $T_{i,1} = 1$). Comme dans la preuve de la proposition 91, on ne cherchera pas ici à simplifier un produit d'arbres en supprimant les éventuels arbres vides. Il y a $n+1$ situations possibles au niveau de la racine de T : pour $1 \leq i \leq n$,

$$\overbrace{B^+(\dots B^+}^{n-i \text{ fois}}(B^-(T_1 \dots T_i)T_{i+1,1} \dots T_{i+1,m_{i+1}}) \dots)T_{n,1} \dots T_{n,m_n}) \in \mathbb{T},$$

et $B^+(\dots B^+(\bullet_1 T_{1,1} \dots T_{1,m_1}) \dots)T_{n,1} \dots T_{n,m_n}) \in \mathbb{T}$. Pour deux valeurs distinctes de i, l'indexation des sommets de T sera différente. Fixons un $0 \leq i \leq n$. Par hypothèse de récurrence, il y a $a_{R_{T_j}}$ façons d'indexer T_j en un élément de $\mathbb{T}$, pour $1 \leq j \leq i$; et $a_{R_{T_{j,l}}}$ façons d'indexer $T_{j,l}$ en un élément de $\mathbb{T}$, pour $i+1 \leq j \leq n$ et $1 \leq l \leq m_j$ (si $T_{j,l}$ est différent de l'arbre vide). Au total, il y a bien

$$a_{R_T} = \prod_{v'\to r} a_{v'} + \sum_{v'\to r} \left(\prod_{v''\to r \text{ tq } v''<v'} a_{v''} \right) \left(\prod_{v''\to w\to r \text{ tq } v'\leq w} a_{v''} \right)$$

façons d'indexer les sommets de T pour obtenir un élément de $\mathbb{T}$. $\square$

Définissons l'algèbre $\mathbf{B}$ comme l'algèbre librement engendrée par l'ensemble $\mathbb{T} \cup \{1\}$. On a alors la propriété remarquable suivante :

Proposition 99 *L'algèbre $\mathbf{B}$ est une algèbre de Hopf.*

Preuve. Montrons que, en réalisant une coupe simple d'un arbre appartenant à $\mathbb{T}$, la branche et le tronc appartiennent respectivement à $\mathbb{T}$ et $\mathbb{T} \cup \{1\}$. On travaille par récurrence sur le degré des arbres. Le résultat est trivial pour $n = 1, 2$ et 3. Au rang $n \geq 4$. Considérons un arbre $T \in \mathbb{T}$ de degré n et $\boldsymbol{v} \models V(T)$ une coupe simple. Il y a deux cas :

1. Si l'arbre est de la forme $T = B^-(T_1 \dots T_m) \in \mathbb{T}^{(n,-)}$, avec $m \geq 1$. Par construction, $T_1, \dots, T_m \in \mathbb{T}$. Si $\boldsymbol{v}$ est la coupe totale, le résultat est évident. Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (on inclut le cas de la coupe totale qui correspond à couper l'arête entre la racine de T et celle de T_i). Par récurrence, comme $T_i \in \mathbb{T}$, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{T}$. De même, par récurrence $Roo_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{T} \cup \{1\}$ donc la forêt $T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m$ est constituée d'arbres appartenant à $\mathbb{T} \cup \{1\}$ et ainsi $Roo_{\boldsymbol{v}}(T) = B^-(T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m) \in \mathbb{T}$.
2. Si l'arbre est de la forme $T = B^+(T_1 \dots T_m) \in \mathbb{T}^{(n,+)}$, avec $m \geq 1$. Comme précédemment, $T_1, \dots, T_m \in \mathbb{T}$. Alors :
   (a) Si $\boldsymbol{v}$ est la coupe simple correspondant à couper l'arête joignant la racine de T (qui est la racine de T_1) et le sommet indexé par n joignant les racines communes de $T_2 \dots T_m$, alors $Roo_{\boldsymbol{v}}(T) = T_1 \in \mathbb{T}$ et $Lea_{\boldsymbol{v}}(T) = B^-(T_2 \dots T_m) \in \mathbb{T}$. Le résultat est donc démontré dans ce cas.
   (b) Sinon, comme $\boldsymbol{v}$ est une coupe simple de T, il existe un unique $i \in \{1, \dots, m\}$ tel que $\boldsymbol{v} \models V(T_i)$ (on inclut le cas de la coupe totale qui, si $i \geq 2$, correspond à couper l'arête entre le sommet de T indexé par n et la racine de T_j). Deux cas sont à distinguer :
      i. Si $i = 1$. Si $\boldsymbol{v} \models V(T_1)$ est la coupe totale, alors $Roo_{\boldsymbol{v}}(T) = 1$ et $Lea_{\boldsymbol{v}}(T) = T$ et le résultat est trivial. Sinon, par récurrence, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_1) \in \mathbb{T}$ et $Roo_{\boldsymbol{v}}(T_1) \in \mathbb{T}$, donc $Roo_{\boldsymbol{v}}(T) = B^+(Roo_{\boldsymbol{v}}(T_1) T_2 \dots T_m) \in \mathbb{T}$.
      ii. Si $i \geq 2$. Par récurrence, $Lea_{\boldsymbol{v}}(T) = Lea_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{T}$ et $Roo_{\boldsymbol{v}}(T_i)$ appartient à $\mathbb{T} \cup \{1\}$. Donc $Roo_{\boldsymbol{v}}(T) = B^+(T_1 \dots Roo_{\boldsymbol{v}}(T_i) \dots T_m) \in \mathbb{T}$.

Ainsi, dans tous les cas, $Roo_{\boldsymbol{v}}(T) \in \mathbb{T} \cup \{1\}$ et $Lea_{\boldsymbol{v}}(T) \in \mathbb{T}$, et on peut conclure par le principe de récurrence. □

Remarque. Les relations d'inclusion suivantes sont évidement vérifiées :

$$\mathbf{B}^0 \subseteq \dots \subseteq \mathbf{B}^i \subseteq \dots \subseteq \mathbf{B}^\infty \subseteq \mathbf{B}.$$

Rappelons que $\mathbf{B}^0$ est la sous-algèbre de $\mathbf{B}$ engendrée par les arbres construits uniquement avec des B^+. De la même façon, on peut définir une sous-algèbre de $\mathbf{B}$ en considérant la sous-algèbre engendrée par les arbres de $\mathbf{B}$ qui sont construits uniquement avec des B^-. Notons-la $\mathbf{B}_l$ (cette terminologie est justifiée par le théorème 112). Elle est clairement stable par coupe admissible, c'est donc une algèbre de Hopf. Il existe un isomorphisme d'algèbres de Hopf entre $\mathbf{B}_l$ et $\mathbf{H}_{NCK}$: à chaque arbre de $\mathbf{H}_{NCK}$ il y a une seule et unique façon de numéroter les sommets (en numérotant les sommets dans la direction "sud-est") ; cela définit une bijection entre les arbres de $\mathbf{B}_l$ et les arbres de $\mathbf{H}_{NCK}$ qui s'étend en un isomorphisme d'algèbres graduées respectant le coproduit.

Il est possible de calculer la série formelle de l'algèbre $\mathbf{B}$:

Proposition 100 *La série formelle de l'algèbre de Hopf* $\mathbf{B}$ *est donnée par la formule :*

$$F_{\mathbf{B}}(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}.$$

Preuve. Notons $f_n^{\mathbf{B}}$ le nombre de forêts de degré n de $\mathbf{B}$ et $f_{1,n}^{\mathbf{B}}$ le nombre d'arbres de degré n. On déduit de la construction de $\mathbf{B}$ les relations suivantes :

$$\begin{array}{rcl} f_1^{\mathbf{B}} & = & 1 \\ f_{1,1}^{\mathbf{B}} & = & 1 \\ f_{1,n}^{\mathbf{B}} & = & 2 f_{n-1}^{\mathbf{B}} \text{ si } n \geq 2. \end{array}$$

Introduisons la convention suivante : $f_{1,0}^{\mathbf{B}} = 0$ et $f_0^{\mathbf{B}} = 1$. On pose alors $F_{\mathbf{B}}(x) = \sum_{n \geq 0} f_n^{\mathbf{B}} x^n$ et $T_{\mathbf{B}}(x) = \sum_{n \geq 0} f_{1,n}^{\mathbf{B}} x^n$. L'algèbre étant libre,

$$F_{\mathbf{B}}(x) = \frac{1}{1 - T_{\mathbf{B}}(x)}.$$

Alors :

$$T_{\mathbf{B}}(x) - x = \sum_{n \geq 2} f_{1,n}^{\mathbf{B}} x^n = 2x \left(\sum_{n \geq 1} f_n^{\mathbf{B}} x^n \right) = 2x \left(\frac{1}{1 - T_{\mathbf{B}}(x)} - 1 \right)$$

Donc : $T_{\mathbf{B}}^2(x)+(x-1)T_{\mathbf{B}}(x)+x=0$. Comme $f_{1,0}^{\mathbf{B}}=0$,

$$T_{\mathbf{B}}(x)=\frac{1-x-\sqrt{1-6x+x^2}}{2} \quad \text{et} \quad F_{\mathbf{B}}(x)=\frac{1+x-\sqrt{1-6x+x^2}}{4x}.$$

□

Voici quelques valeurs numériques :

k	1	2	3	4	5	6	7	8
$f_{1,k}^{\mathbf{B}}$	1	2	6	22	90	394	1806	8558
$f_k^{\mathbf{B}}$	1	3	11	45	197	903	4279	20793

Ce sont les séquences A006318 et A001003 de [Slo].

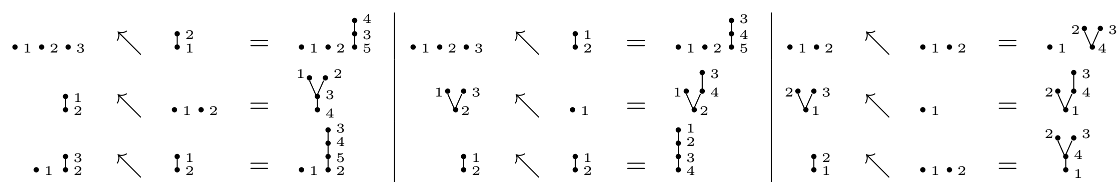
4.2.2 Coliberté de B

Introduisons une nouvelle opération sur $\mathbf{B}$:

Étant données deux forêts non vides F et G appartenant à $\mathbf{B}$, on définit une forêt $F \nwarrow G$ en greffant à la feuille la plus à droite de F la forêt G et en indexant les sommets comme suit : on laisse les indices de F inferieurs strictement à l'indice de sa feuille la plus à droite invariants ; on numérote ensuite la forêt G en préservant l'ordre d'origine de ses indices ; on finit alors en numérotant le reste des sommets non encore indexés de F en préservant ici encore l'ordre d'origine des indices dans F. Si F est une forêt de $\mathbf{B}$, on pose $1 \nwarrow F = F \nwarrow 1 = F$. Par linéarité, on définit ainsi une nouvelle opération $\nwarrow: \mathbf{B}\times\mathbf{B}\rightarrow\mathbf{B}$.

Remarque. Dans le cas particulier où $F = \bullet_1$, $\bullet_1 \nwarrow G = B^-(G_1\dots G_n)$, pour toute forêt $G = G_1\dots G_n \in \mathbf{B}$.

Exemples. On illustre ci-dessous l'opération $\nwarrow$:



Il faut montrer que $\mathbf{B}$ est stable par l'opération de greffe $\nwarrow$ ainsi définie. Par définition de la greffe, il suffit de montrer le résultat lorsque F est un arbre non vide et $G = G_1\dots G_n$ une forêt non vide de $\mathbf{B}$. Comme $\mathbf{B}$ est librement engendrée par $\mathbb{T}\cup\{1\}$ en tant qu'algèbre, $G_1,\dots,G_n\in\mathbb{T}$. Si F est l'arbre constitué d'un unique sommet, alors $F\nwarrow G = B^-(G_1\dots G_n)\in\mathbb{T}\subseteq\mathbf{B}$. Supposons maintenant F de degré ≥ 2. Alors, la branche de F sur laquelle on greffe G comporte au moins deux sommets. Il y a deux cas à distinguer :

1. Si l'indice de la feuille la plus à droite de F (celle où on greffe) est plus grand que l'indice de son père. Dans ce cas, cette feuille a été insérée lors de la construction de F par un $B^+(\dots,1)$. Pour obtenir l'arbre $F\nwarrow G$, il suffit alors de reproduire les opérations faites pour construire F à un changement près, en remplaçant $B^+(\dots 1)$ par $B^+(\dots G_1\dots G_n)$. Ainsi, $F\nwarrow G$ appartient à $\mathbb{T}\subseteq\mathbf{B}$.
2. Si l'indice de la feuille la plus à droite de F est plus petit que l'indice de son père. Dans ce cas, cette feuille a été insérée lors de la construction de F par un $B^-(H_1\dots H_k\bullet_1)$ où $H_1,\dots,H_k,\bullet_1$ est la suite des sous-arbres issus du même sommet de F, le père de la feuille la plus à droite de F. Par construction, $H_1,\dots,H_k\in\mathbb{T}$. Pour obtenir l'arbre $F\nwarrow G$, il suffit alors de reproduire les opérations faites pour construire F à un changement près, en remplaçant $B^-(H_1\dots H_k\bullet_1)$ par $B^-(H_1\dots H_k B^-(G_1\dots G_n))$. Donc $F\nwarrow G$ appartient aussi à $\mathbb{T}\subseteq\mathbf{B}$ dans ce cas.

Le lemme qui suit est évident :

Lemme 101 *Pour toutes forêts $F,G,H\in\mathbf{B}$,*

$$\begin{aligned}(F\nwarrow G)\nwarrow H &= F\nwarrow(G\nwarrow H),\\ (FG)\nwarrow H &= F(G\nwarrow H).\end{aligned}$$

Rappelons la définition suivante (voir par exemple [Foi12, Lod08]) :

Définition 102 *Une algèbre dupliciale est un triplet $(A, *, \nwarrow)$, où A est un espace vectoriel et $*, \nwarrow$: $A \otimes A \longrightarrow A$, avec les axiomes suivants : pour tout $x, y, z \in A$,*

$$\left\{\begin{array}{rcl} (x*y)*z &=& x*(y*z), \\ (x \nwarrow y) \nwarrow z &=& x \nwarrow (y \nwarrow z), \\ (x*y) \nwarrow z &=& x*(y \nwarrow z). \end{array}\right.$$

Ainsi, $(\mathbf{B})_+$ munit de la concaténation et de $\nwarrow$ est une algèbre dupliciale.

Proposition 103 $(\mathbf{B}_l)_+$ *est l'algèbre dupliciale libre générée par l'élément* $\bullet_1$.

Preuve. Soit A une algèbre dupliciale et soit $a \in A$. Il s'agit de montrer qu'il existe un unique morphisme d'algèbres dupliciales $\phi : (\mathbf{B}_l)_+ \to A$ tel que $\phi(\bullet_1) = a$. On définit $\phi(F)$ pour toute forêt non vide $F \in (\mathbf{B}_l)_+$ inductivement sur le degré de F :

$$\begin{array}{rcl} \phi(\bullet_1) &=& a, \\ \phi(T_1 \dots T_k) &=& \phi(T_1)\dots\phi(T_k) \text{ si } k \geq 2, \\ \phi(B^-(T_1 \dots T_k)) &=& a \nwarrow \phi(T_1 \dots T_k) \text{ si } k \geq 1, \end{array}$$

avec $T_1, \dots, T_k \in \mathbb{T} \cap \mathbf{B}_l$. Comme le produit dans A est associatif, ϕ est bien définie. ϕ s'étend par linéarité en une application $\phi : (\mathbf{B}_l)_+ \to A$. Montrons que c'est un morphisme d'algèbres dupliciales. Par le second point, $\phi(FG) = \phi(F)\phi(G)$ pour toutes forêts $F, G \in (\mathbf{B}_l)_+$. Il reste à prouver que $\phi(F \nwarrow G) = \phi(F) \nwarrow \phi(G)$ pour toutes forêts $F, G \in (\mathbf{B}_l)_+$. On pose $F = F_1 \dots F_k$, avec $k \geq 1$ et $F_1, \dots, F_k \in \mathbb{T} \cap \mathbf{B}_l$. Par récurrence sur le degré n de F_k. Si $n = 1$, alors $F_k = \bullet_1$, et, en notant $G = G_1 \dots G_m$, avec $m \geq 1$ et $G_1, \dots, G_m \in \mathbb{T} \cap \mathbf{B}_l$:

$$\begin{array}{rcl} \phi(F \nwarrow G) &=& \phi((F_1 \dots F_{k-1} \bullet_1) \nwarrow G) \\ &=& \phi(F_1 \dots F_{k-1} B^-(G_1 \dots G_m)) \\ &=& \phi(F_1 \dots F_{k-1})\phi(B^-(G_1 \dots G_m)) \\ &=& \phi(F_1 \dots F_{k-1})(a \nwarrow \phi(G_1 \dots G_m)) \\ &=& \phi(F_1 \dots F_{k-1})(\phi(F_k) \nwarrow \phi(G)) \\ &=& \phi(F) \nwarrow \phi(G). \end{array}$$

Soit $n \geq 2$ et supposons le résultat vérifié si le degré de F_k est strictement inferieur à n. $F_k \in \mathbb{T} \cap \mathbf{B}_l$, donc F_k est de la forme $F_k = B^-(H_1 \dots H_l)$ avec $l \geq 1$ et $H_1, \dots, H_l \in \mathbb{T} \cap \mathbf{B}_l$. Alors, en utilisant l'hypothèse de récurrence sur la forêt $H = H_1 \dots H_l$:

$$\begin{array}{rcl} \phi(F \nwarrow G) &=& \phi(F_1 \dots F_{k-1} B^-(H_1 \dots H_l) \nwarrow G) \\ &=& \phi(F_1 \dots F_{k-1})\phi(B^-(H_1 \dots H_l \nwarrow G)) \\ &=& \phi(F_1 \dots F_{k-1})(a \nwarrow \phi(H \nwarrow G)) \\ &=& \phi(F_1)\dots\phi(F_{k-1})(a \nwarrow (\phi(H) \nwarrow \phi(G))) \\ &=& \phi(F_1)\dots\phi(F_{k-1})((a \nwarrow \phi(H)) \nwarrow \phi(G)) \\ &=& \phi(F_1)\dots\phi(F_{k-1})(\phi(F_k) \nwarrow \phi(G)) \\ &=& (\phi(F_1)\dots\phi(F_{k-1})\phi(F_k)) \nwarrow \phi(G) \\ &=& \phi(F) \nwarrow \phi(G)). \end{array}$$

Ainsi ϕ est un morphisme d'algèbres dupliciales.

Soit $\phi' : (\mathbf{B}_l)_+ \to A$ un deuxième morphisme d'algèbres dupliciales tel que $\phi'(\bullet_1) = a$. Pour tout arbre non vide $T_1, \dots, T_k \in \mathbb{T} \cap \mathbf{B}_l$, $\phi'(T_1 \dots T_k) = \phi'(T_1)\dots\phi'(T_k)$ et $\phi'(B^-(T_1 \dots T_k)) = \phi'(\bullet_1 \nwarrow (T_1 \dots T_k)) = a \nwarrow \phi'(T_1 \dots T_k)$. Donc $\phi = \phi'$. □

Définition 104 *Pour toute forêt non vide $F \in (\mathbf{B})_+$, notons L_F^r la feuille la plus à droite de F. On pose :*

$$\begin{array}{rcl} \tilde{\Delta}_{\prec}(F) &=& \displaystyle\sum_{\boldsymbol{v} \models V(F) \text{ et } L_F^r \in Lea_{\boldsymbol{v}}(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F), \\ \tilde{\Delta}_{\succ}(F) &=& \displaystyle\sum_{\boldsymbol{v} \models V(F) \text{ et } L_F^r \in Roo_{\boldsymbol{v}}(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F). \end{array}$$

Remarquons que $\tilde{\Delta}_\prec + \tilde{\Delta}_\succ = \tilde{\Delta}$.

Lemme 105 *Pour tout $F \in (\mathbf{B})_+$,*

$$\left\{\begin{array}{rcl}(\tilde{\Delta}_\prec \otimes Id)\circ\tilde{\Delta}_\prec(F) &=& (Id\otimes\tilde{\Delta})\circ\tilde{\Delta}_\prec(F),\\ (\tilde{\Delta}_\succ \otimes Id)\circ\tilde{\Delta}_\prec(F) &=& (Id\otimes\tilde{\Delta}_\succ)\circ\tilde{\Delta}_\prec(F),\\ (\tilde{\Delta} \otimes Id)\circ\tilde{\Delta}_\succ(F) &=& (Id\otimes\tilde{\Delta}_\succ)\circ\tilde{\Delta}_\succ(F).\end{array}\right. \quad (4.4)$$

En d'autres termes, $(\mathbf{B})_+$ est une coalgèbre dendriforme.

Preuve. Soit F une forêt non vide de $\mathbf{B}$. Par coassociativité de $\tilde{\Delta}$, on peut poser :

$$(\tilde{\Delta}\otimes Id)\circ\tilde{\Delta}(F) = (Id\otimes\tilde{\Delta})\circ\tilde{\Delta}(F) = \sum F^{(1)}\otimes F^{(2)}\otimes F^{(3)},$$

où $F^{(1)}, F^{(2)}, F^{(3)}$ sont des sous-forêts non vides de F. Alors :

$$(\tilde{\Delta}_\prec\otimes Id)\circ\tilde{\Delta}_\prec(F) = (Id\otimes\tilde{\Delta})\circ\tilde{\Delta}_\prec(F) = \sum_{L_F^r\in F^{(1)}} F^{(1)}\otimes F^{(2)}\otimes F^{(3)},$$
$$(\tilde{\Delta}_\succ\otimes Id)\circ\tilde{\Delta}_\prec(F) = (Id\otimes\tilde{\Delta}_\succ)\circ\tilde{\Delta}_\prec(F) = \sum_{L_F^r\in F^{(2)}} F^{(1)}\otimes F^{(2)}\otimes F^{(3)},$$
$$(\tilde{\Delta}\otimes Id)\circ\tilde{\Delta}_\succ(F) = (Id\otimes\tilde{\Delta}_\succ)\circ\tilde{\Delta}_\succ(F) = \sum_{L_F^r\in F^{(3)}} F^{(1)}\otimes F^{(2)}\otimes F^{(3)}.$$

□

Notations. Soit $(A, \tilde{\Delta}_\prec, \tilde{\Delta}_\succ)$ une coalgèbre dendriforme.

1. On notera $Prim_{tot}(A) = Ker(\tilde{\Delta}_\succ)\cap Ker(\tilde{\Delta}_\prec)$.
2. Nous utiliserons les notations de Sweedler suivantes : pour tout $a \in A$,

$$\tilde{\Delta}(a) = a'\otimes a'', \quad \tilde{\Delta}_\prec(a) = a'_\prec\otimes a''_\prec, \quad \tilde{\Delta}_\succ(a) = a'_\succ\otimes a''_\succ$$

Proposition 106 *1. Soit $x, y \in (\mathbf{B})_+$. Alors :*

$$\left\{\begin{array}{rcl}\tilde{\Delta}_\prec(xy) &=& y\otimes x + x'y\otimes x'' + xy'_\prec\otimes y''_\prec + y'_\prec\otimes xy''_\prec + x'y'_\prec\otimes x''y''_\prec,\\ \tilde{\Delta}_\succ(xy) &=& x\otimes y + x'\otimes x''y + xy'_\succ\otimes y''_\succ + y'_\succ\otimes xy''_\succ + x'y'_\succ\otimes x''y''_\succ.\end{array}\right. \quad (4.5)$$

En d'autres termes, $(\mathbf{B})_+$ est une bialgèbre codendriforme.

2. Soit $x, y \in (\mathbf{B})_+$. Alors :

$$\left\{\begin{array}{rcl}\tilde{\Delta}_\prec(x\nwarrow y) &=& y\otimes x + y'_\prec\otimes x\nwarrow y''_\prec + x'_\prec\nwarrow y\otimes x''_\prec + x'_\succ y\otimes x_\succ\\ && +x'_\succ y'_\prec\otimes x''_\succ\nwarrow y''_\prec,\\ \tilde{\Delta}_\succ(x\nwarrow y) &=& y'_\succ\otimes x\nwarrow y''_\succ + x'_\succ\otimes x''_\succ\nwarrow y + x'_\succ y'_\succ\otimes x''_\succ\nwarrow y''_\succ.\end{array}\right. \quad (4.6)$$

Preuve. Il suffit de prouver ces formules pour $x = F$, $y = G$ des forêts non vides $\in \mathbb{T}$. Commençons par prouver les premières formules, à savoir $\tilde{\Delta}_\prec(FG)$ et $\tilde{\Delta}_\succ(FG)$. Pour toute coupe admissible $\boldsymbol{v} \models\!\!\!| V(FG)$, soit $\boldsymbol{v}'$ la restriction de $\boldsymbol{v}$ à F et $\boldsymbol{v}''$ la restriction de $\boldsymbol{v}$ à G. Alors $\boldsymbol{v}' \models V(F)$ et $\boldsymbol{v}'' \models V(G)$. Par ailleurs, $\boldsymbol{v}'$ et $\boldsymbol{v}''$ ne sont pas simultanément totales, ni simultanément vides.

Pour $\tilde{\Delta}_\prec(FG)$: soit $\boldsymbol{v} \models\!\!\!| V(FG)$, telle que $R_{FG} = R_G$ appartient à $Lea_{\boldsymbol{v}}(FG)$. Comme $R_G \in Lea_{\boldsymbol{v}''}(G)$, $\boldsymbol{v}''$ est non vide. On a alors cinq possibilités pour $\boldsymbol{v}$:

- $\boldsymbol{v}'$ est vide et $\boldsymbol{v}''$ est totale : cela donne le terme $G\otimes F$.
- $\boldsymbol{v}'$ est non vide et $\boldsymbol{v}''$ est totale : alors $\boldsymbol{v}' \models\!\!\!| V(F)$, et cela donne le terme $F'G\otimes F''$.
- $\boldsymbol{v}'$ est vide et $\boldsymbol{v}''$ est non totale : comme $R_G \in Lea_{\boldsymbol{v}''}(G)$, cela donne le terme $G'_\prec\otimes FG''_\prec$.
- $\boldsymbol{v}'$ est totale et $\boldsymbol{v}''$ est non totale : comme $R_G \in Lea_{\boldsymbol{v}''}(G)$, cela donne le terme $FG'_\prec\otimes G''_\prec$.
- $\boldsymbol{v}' \models\!\!\!| V(F)$ et $\boldsymbol{v}''$ sont non totales : comme $R_G \in Lea_{\boldsymbol{v}''}(G)$, cela donne le terme $F'G'_\prec\otimes F''G''_\prec$.

Intérressons nous à présent à $\tilde{\Delta}_\succ(FG)$. Soit $\boldsymbol{v} \models\!\!\!| V(FG)$, telle que $R_{FG} = R_G$ appartient à $Roo_{\boldsymbol{v}}(FG)$. Comme $R_G \in Roo_{\boldsymbol{v}''}(G)$, $\boldsymbol{v}''$ n'est pas totale. Il y a cinq possibilités pour $\boldsymbol{v}$:

- $\boldsymbol{v}'$ est totale et $\boldsymbol{v}''$ est vide : cela donne le terme $F\otimes G$.
- $\boldsymbol{v}'$ est non totale et $\boldsymbol{v}''$ est vide : alors $\boldsymbol{v}' \models\!\!\!| V(F)$, et cela donne le terme $F'\otimes F''G$.

– $\boldsymbol{v}'$ est totale et $\boldsymbol{v}''$ est non vide : comme $R_G \in Roo_{\boldsymbol{v}''}(G)$, cela donne le terme $FG'_{\succ} \otimes G''_{\succ}$.
– $\boldsymbol{v}'$ est vide et $\boldsymbol{v}''$ est non totale : comme $R_G \in Roo_{\boldsymbol{v}''}(G)$, cela donne le terme $G'_{\succ} \otimes FG''_{\succ}$.
– $\boldsymbol{v}' \models V(F)$ et $\boldsymbol{v}''$ sont non totales : comme $R_G \in Roo_{\boldsymbol{v}''}(G)$, cela donne le terme $F'G'_{\succ} \otimes F''G''_{\succ}$.

Pour une coupe admissible $\boldsymbol{v} \models V(F \nwarrow G)$, soit $\boldsymbol{v}'$ la restriction à $\boldsymbol{v}$ de F et soit $\boldsymbol{v}''$ l'unique coupe admissible de G telle que $Lea_{\boldsymbol{v}''}(G)$ est la sous-forêt de $Lea_{\boldsymbol{v}}(F \nwarrow G)$ formée par les sommets qui appartiennent à $V(G)$. Remarquons que, $\boldsymbol{v}$ n'étant pas totale, $\boldsymbol{v}'$ n'est pas totale.

Calculons $\tilde{\Delta}_{\prec}(F \nwarrow G)$. Soit $\boldsymbol{v} \models V(F \nwarrow G)$, telle que $R_{F \nwarrow G} = R_G$ appartient à $Lea_{\boldsymbol{v}}(F \nwarrow G)$. Comme $R_G \in Lea_{\boldsymbol{v}''}(G)$, $\boldsymbol{v}''$ est non vide. Il y a quatre possibilités pour $\boldsymbol{v}$:
– $\boldsymbol{v}'$ est vide et $\boldsymbol{v}''$ est totale : cela donne le terme $G \otimes F$.
– $\boldsymbol{v}'$ est vide et $\boldsymbol{v}''$ est non totale : alors $\boldsymbol{v}'' \models V(G)$ et $R_G \in Lea_{\boldsymbol{v}''}(G)$, et cela donne le terme $G'_{\prec} \otimes F \nwarrow G''_{\prec}$.
– $\boldsymbol{v}'$ est non vide et $\boldsymbol{v}''$ est totale : on a alors deux sous-cas :
  – $Lea_{\boldsymbol{v}'}(F)$ contient R_F : cela donne le terme $F'_{\prec} \nwarrow G \otimes F''_{\prec}$.
  – $Roo_{\boldsymbol{v}'}(F)$ contient R_F : cela donne le terme $F'_{\succ} G \otimes F''_{\succ}$.
– $\boldsymbol{v}'$ est non vide et $\boldsymbol{v}''$ est non totale : alors R_F n'appartient pas à $Lea_{\boldsymbol{v}'}(F)$, R_G appartient à $Lea_{\boldsymbol{v}''}(G)$, et cela donne le terme $F'_{\succ} G'_{\prec} \otimes F''_{\succ} \nwarrow G''_{\prec}$.

Pour terminer, il reste à calculer $\tilde{\Delta}_{\succ}(F \nwarrow G)$. Soit $\boldsymbol{v} \models V(F \nwarrow G)$, telle que $R_{F \nwarrow G} = R_G$ appartient à $Roo_{\boldsymbol{v}}(F \nwarrow G)$. Comme $R_G \in Roo_{\boldsymbol{v}''}(G)$, $\boldsymbol{v}''$ n'est pas totale. Donc $\boldsymbol{v}'$ ne contient pas R_F. Il y a trois possibilités pour $\boldsymbol{v}$:
– $\boldsymbol{v}'$ est vide : alors $\boldsymbol{v}'' \models V(G)$ et $Roo_{\boldsymbol{v}''}(G)$ contient R_G, et cela donne le terme $G'_{\succ} \otimes F \nwarrow G''_{\succ}$.
– $\boldsymbol{v}'$ est non vide et $\boldsymbol{v}''$ est vide : alors $R_F \in Roo_{\boldsymbol{v}'}(F)$ et on obtient le terme $F'_{\succ} \otimes F''_{\succ} \nwarrow G$.
– $\boldsymbol{v}'$ est non vide et $\boldsymbol{v}''$ est non vide : alors $R_F \in Roo_{\boldsymbol{v}'}(F)$, $R_G \in Roo_{\boldsymbol{v}''}(G)$ et on obtient le terme $F'_{\succ} G'_{\succ} \otimes F''_{\succ} \nwarrow G'_{\succ}$.

□

Rappelons la définition suivante :

Définition 107 *Une bialgèbre dupliciale dendriforme est une famille $(A, *, \nwarrow, \tilde{\Delta}_{\prec}, \tilde{\Delta}_{\succ})$, où A est un espace vectoriel, $*, \nwarrow : A \otimes A \longrightarrow A$ et $\tilde{\Delta}_{\prec}, \tilde{\Delta}_{\succ} : A \longrightarrow A \otimes A$, avec les propriétés suivantes :*

1. *$(A, *, \nwarrow)$ est une algèbre dupliciale (axiomes (4.4)).*
2. *$(A, \tilde{\Delta}_{\prec}, \tilde{\Delta}_{\succ})$ est une coalgèbre dendriforme (axiomes (4.4)).*
3. *Les compatibilités de la proposition 106 sont satisfaites (axiomes (4.5) et (4.6)).*

Nous avons besoin du théorème de rigidité suivant, démontré dans [Foi12],

Théorème 108 *Soit A une bialgèbre dupliciale dendriforme. On suppose que A est graduée et connexe, c'est-à-dire que $A_0 = (0)$. Soit $(p_d)_{d \in \mathcal{D}}$ une base de $Prim_{tot}(A)$ formée par des éléments homogènes, indexés par un ensemble gradué $\mathcal{D}$. Il existe un unique isomorphisme de bialgèbres dupliciales dendriformes graduées :*

$$\phi : \left\{ \begin{array}{rcl} (\mathbf{H}^{\mathcal{D}}_{NCK})_+ & \longrightarrow & A \\ \bullet_d,\ d \in \mathcal{D} & \longrightarrow & p_d. \end{array} \right.$$

On en déduit alors le résultat suivant :

Théorème 109 *Il existe un ensemble gradué $\mathcal{D}$ tel que $(\mathbf{B})_+$ est isomorphe à $(\mathbf{H}^{\mathcal{D}}_{NCK})_+$ comme bialgèbres dupliciales dendriformes graduées.*

La série formelle de $\mathcal{D}$ est donnée par :

$$f_{\mathcal{D}}(x) = \frac{f_{\mathbf{B}}(x) - 1}{f_{\mathbf{B}}(x)^2}.$$

Cela donne :

k	1	2	3	4	5	6	7	8
$\lvert\mathcal{D}\rvert_v$	1	1	2	6	22	90	394	1806

On retrouve les nombres de Schroeder, correspondants à la séquence A006318 de [Slo].

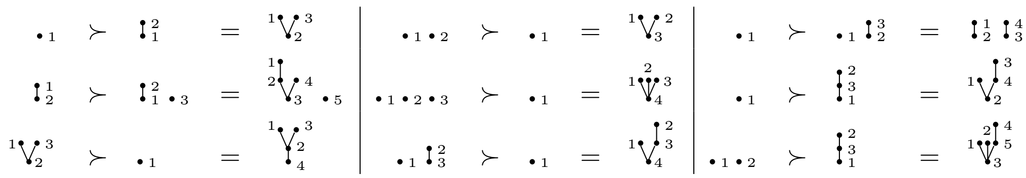
Corollaire 110 *L'algèbre de Hopf **B** est libre (ce qu'on savait déjà, librement engendrée par $\mathbb{T}$), colibre et auto-duale.*

4.2.3 Structure d'algèbre bigreffe de B

Définissons à présent une nouvelle loi de composition interne sur $(\mathbf{B})_+$ qui va permettre de munir $\mathbf{B}$ d'une structure d'algèbre de greffes à gauche unitaire.

Étant donnés deux arbres non vides T et G de $\mathbb{T}$, on définit un arbre $T \succ G$ en greffant par la gauche T sur la racine de G et en indexant les sommets comme suit : on conserve l'indexation des sommets de T et on numérote ensuite les sommets de G dans leurs ordres de départ mais en décalant leurs indices par le nombre de sommets de T. Considérons maintenant une forêt $T_1 \ldots T_n$ et un arbre G, avec $n \geq 1$ et $T_1, \ldots, T_n, G \in \mathbb{T}$ (tous non vides), on définit l'arbre $(T_1 \ldots T_n) \succ G$ en le posant égal à $T_1 \succ (T_2 \succ (\ldots (T_n \succ G) \ldots))$. Étant données deux forêts non vides $T_1 \ldots T_n$ et $G_1 \ldots G_m$, avec $n, m \geq 1$ et $T_1, \ldots, T_n, G_1, \ldots, G_m \in \mathbb{T}$, on pose $(T_1 \ldots T_n) \succ (G_1 \ldots G_m) = ((T_1 \ldots T_n) \succ G_1) G_2 \ldots G_m$. En étendant par linéarité $\succ$, on définit ainsi une nouvelle opération sur $(\mathbf{B})_+$. On utilisera la convention suivante : si $T \in (\mathbf{B})_+$, $1 \succ T = T$ et $T \succ 1 = 0$.

Exemples. Illustrons ci-dessous l'opération $\succ$:



Montrons que $(\mathbf{B})_+$ est bien stable pour la loi $\succ$. Il suffit de voir que si T et G sont deux arbres $\in \mathbb{T}$, l'arbre $T \succ G$ définit précédemment est encore un élément de $\mathbb{T}$. Pour cela, notons $G_1, \ldots, G_n$ la suite des sous-arbres issus de la racine de G, et $G_{i,1}, \ldots, G_{i,m_i}$ la suite des sous-arbres issus de la racine de G_i, pour tout $i \in \{1, \ldots, n\}$. Il y a alors deux cas :

1. Si $G = B^+(\ldots B^+(\bullet_1 G_{1,1} \ldots G_{1,m_1}) \ldots) G_{n,1} \ldots G_{n,m_n})$, alors par définition de $\succ$,
$$T \succ G = B^+(\ldots B^+(B^-(T) G_{1,1} \ldots G_{1,m_1}) \ldots) G_{n,1} \ldots G_{n,m_n})$$
et ceci est bien un élément de $\mathbb{T}$.
2. Si il existe un $1 \leq i \leq n$ tel que
$$G = \overbrace{B^+(\ldots B^+}^{n-i \text{ fois}}(B^-(G_1 \ldots G_i) G_{i+1,1} \ldots G_{i+1,m_{i+1}}) \ldots) G_{n,1} \ldots G_{n,m_n}),$$
alors par définition de $\succ$,
$$T \succ G = \overbrace{B^+(\ldots B^+}^{n-i \text{ fois}}(B^-(T G_1 \ldots G_i) G_{i+1,1} \ldots G_{i+1,m_{i+1}}) \ldots) G_{n,1} \ldots G_{n,m_n}),$$
et ceci est ici encore un élément de $\mathbb{T}$.

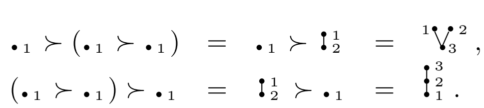
Ainsi $(\mathbf{B})_+$ est stable pour l'opération $\succ$.

Remarque. Pour toute forêt non vide $T_1 \ldots T_n \in (\mathbf{B})_+$, $(T_1 \ldots T_n) \succ \bullet_1 = B^-(T_1 \ldots T_n)$.

La propriété suivante vient directement de la définition de $\succ$:

Lemme 111 *Étant donné $F, G, H \in (\mathbf{B})_+$,*
$$(FG) \succ H = F \succ (G \succ H), \tag{4.7}$$
$$(F \succ G) H = F \succ (GH). \tag{4.8}$$

Remarque. L'opération $\succ$ n'est pas associative. Par exemple,



Grâce au lemme 111, $(\mathbf{B})_+$ muni de la concaténation et de $\succ$ est une algèbre de greffes à gauche.

Théorème 112 $(\mathbf{B}_l)_+$ *est l'algèbre de greffes à gauche libre engendrée par l'élément* $\bullet_1$.

Preuve. Soit A une algèbre de greffes à gauche et soit $a \in A$. Il faut montrer qu'il existe un unique morphisme d'algèbres de greffes à gauche $\phi : (\mathbf{B}_l)_+ \to A$ tel que $\phi(\bullet_1) = a$. On définit $\phi(F)$ pour toute forêt non vide $F \in (\mathbf{B}_l)_+$ inductivement sur le degré de F :

$$\begin{array}{rcl} \phi(\bullet_1) & = & a, \\ \phi(T_1 \dots T_k) & = & \phi(T_1)\dots\phi(T_k) \text{ si } k \geq 2, \\ \phi(B^-(T_1 \dots T_k)) & = & \phi(T_1 \dots T_k) \succ a \text{ si } k \geq 1. \end{array}$$

Comme le produit $*$ dans A est associatif, ϕ est bien définie. ϕ s'étend par linéarité en une application $\phi : (\mathbf{B})_+ \to A$. Montrons que c'est un morphisme d'algèbres de greffes à gauche. Par le second point, $\phi(FG) = \phi(F)\phi(G)$ pour toutes forêts $F, G \in (\mathbf{B}_l)_+$. Considérons deux forêts non vides F et G. Il faut prouver que $\phi(F \succ G) = \phi(F) \succ \phi(G)$. On travaille par induction sur le degré n de G. Si $n = 1$, $G = \bullet_1$, et :

$$\phi(F \succ G) = \phi(B^-(F)) = \phi(F) \succ a = \phi(F) \succ \phi(G).$$

Supposons le résultat vérifié pour toutes forêts de degré $< n$. Considérons alors $G \in (\mathbf{B}_l)_+$ une forêt de degré $n \geq 2$ et une forêt non vide $F = F_1 \dots F_m \in (\mathbf{B}_l)_+$. Notons k la longueur de G. Il y a deux cas suivant la longueur k de G :

1. Si $k \geq 2$, $G = G_1 \dots G_k$. Alors

$$\begin{array}{rcl} \phi(F \succ G) & = & \phi((F_1 \dots F_m) \succ (G_1 \dots G_k)) \\ & = & \phi(((F_1 \dots F_m) \succ G_1)G_2 \dots G_k) \\ & = & \phi((F_1 \dots F_m) \succ G_1)\phi(G_2)\dots\phi(G_k) \\ & = & (\phi(F_1 \dots F_m) \succ \phi(G_1))\phi(G_2)\dots\phi(G_k) \\ & = & \phi(F) \succ (\phi(G_1)\phi(G_2)\dots\phi(G_k)) \\ & = & \phi(F) \succ \phi(G), \end{array}$$

   en utilisant l'hypothèse de récurrence à la quatrième égalité.

2. Si $k = 1$, G est un arbre de degré $n \geq 2$, donc $G = B^-(G_1 \dots G_l)$, avec $G_1, \dots, G_l \in (\mathbf{B}_l)_+$ et $l \geq 1$. Alors

$$\begin{array}{rcl} \phi(F \succ G) & = & \phi((F_1 \dots F_m) \succ B^-(G_1 \dots G_l)) \\ & = & \phi(B^-(F_1 \dots F_m G_1 \dots G_l)) \\ & = & \phi(F_1 \dots F_m G_1 \dots G_l) \succ a \\ & = & (\phi(F_1 \dots F_m)\phi(G_1 \dots G_l)) \succ a \\ & = & \phi(F) \succ (\phi(G_1 \dots G_l) \succ a) \\ & = & \phi(F) \succ \phi(B^-(G_1 \dots G_l)) \\ & = & \phi(F) \succ \phi(G). \end{array}$$

Ainsi, dans tous les cas, $\phi(F \succ G) = \phi(F) \succ \phi(G)$. Par le principe de récurrence, cette formule est donc démontrée pour toute forêt $F, G \in (\mathbf{B}_l)_+$.

Soit $\phi' : (\mathbf{B}_l)_+ \to A$ un deuxième morphisme d'algèbres de greffes à gauche tel que $\phi'(\bullet_1) = a$. Soient $k \geq 1$ et $T_1, \dots, T_k \in \mathbb{G} \cap \mathbf{B}_l$. Alors $\phi'(T_1 \dots T_k) = \phi'(T_1)\dots\phi'(T_k)$. De plus,

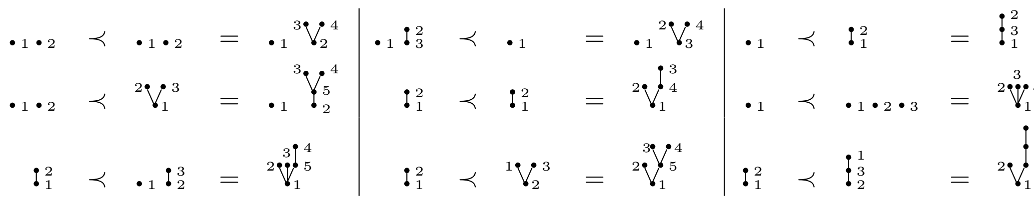
$$\begin{array}{rcl} \phi'(B^-(T_1 \dots T_k)) & = & \phi'((T_1 \dots T_k) \succ \bullet_1) \\ & = & \phi'(T_1 \dots T_k) \succ \phi'(\bullet_1) \\ & = & \phi'(T_1 \dots T_k) \succ a. \end{array}$$

Donc $\phi' = \phi$ et ceci termine la démonstration. □

On peut aussi définir une opération $\prec : (\mathbf{B})_+ \times (\mathbf{B})_+ \to (\mathbf{B})_+$. Cela va permettre de munir $\mathbf{B}$ d'une structure d'algèbre de greffes à droite unitaire.

Étant donnés deux arbres non vides T et G appartenant à $\mathbb{T}$, on définit un arbre $T \prec G$ en greffant par la droite G sur la racine de T et en indexant les sommets comme suit : on conserve l'indexation des sommets de T, puis on numérote les sommets de G différents de la racine de G en conservant l'ordre initial, et on termine en numérotant la racine de G (par $|T|_v + |G|_v$). Considérons maintenant un arbre T et une forêt $G_1 \dots G_m$, avec $T, G_1, \dots, G_m \in \mathbb{T}$, on définit l'arbre $T \prec (G_1 \dots G_m)$ en le posant égal à $(\dots((T \prec G_1) \prec G_2) \dots \prec G_m)$. Étant données deux forêts non vides $T_1 \dots T_n$ et $G_1 \dots G_m$, avec $n, m \geq 1$ et $T_1, \dots, T_n, G_1, \dots, G_m \in \mathbb{T}$, on pose $(T_1 \dots T_n) \prec (G_1 \dots G_m) = T_1 \dots T_{n-1}(T_n \prec (G_1 \dots G_m))$. En étendant par linéarité $\prec$, ceci définit une nouvelle opération sur $(\mathbf{B})_+$. Si $T \in (\mathbf{B})_+$, on pose $T \prec 1 = T$ et $1 \prec T = 0$.

Exemples. Nous illustrons ci-dessous l'opération $\prec$:



Vérifions que $(\mathbf{B})_+$ est bien stable par $\prec$. Il suffit de montrer que si T et G sont deux arbres non vides appartenant à $\mathbb{T}$, l'arbre $T \prec G$ définit précédemment est encore un élément de $\mathbb{T}$. Si $G = \bullet_1$, $T \prec G = B^+(T)$ appartient à $\mathbb{T}$. Supposons maintenant que $|G|_v \geq 2$. On note $G_1, \dots, G_n$ la suite des sous-arbres issus de la racine de G, et $G_{i,1}, \dots, G_{i,m_i}$ la suite des sous-arbres issus de la racine de G_i pour tout $i \in \{1, \dots, n\}$ (si $G_i = \bullet_1$, alors $m_i = 1$ et $G_{i,m_i} = 1$). Il y a alors plusieurs cas :

1. Si $G = B^+(\dots B^+(\bullet_1 G_{1,1} \dots G_{1,m_1}) \dots) G_{n,1} \dots G_{n,m_n})$, où $G_{1,1}, \dots, G_{n,m_n} \in \mathbb{T}$ par construction. Alors, pour tout $i \in \{1, \dots, n\}$, $G_i = B^-(G_{i,1} \dots G_{i,m_i}) \in \mathbb{T}$, et donc
$$T \prec G = B^+(TG_1 \dots G_n)$$
est un élément de $(\mathbf{B})_+$.
2. Si il existe un $1 \leq i \leq n$ tel que
$$G = \overbrace{B^+(\dots B^+}^{n-i \text{ fois}}(B^-(G_1 \dots G_i) G_{i+1,1} \dots G_{i+1,m_{i+1}}) \dots) G_{n,1} \dots G_{n,m_n}),$$
avec par construction $G_1, \dots, G_i, G_{i+1,1}, \dots, G_{n,m_n} \in \mathbb{T}$. Alors, pour tout $i+1 \leq j \leq n$, $G_j = B^-(G_{j,1} \dots G_{j,m_j}) \in \mathbb{T}$, et donc
$$T \prec G = B^+(TG_1 \dots G_i G_{i+1} \dots G_n)$$
est un élément de $(\mathbf{B})_+$.

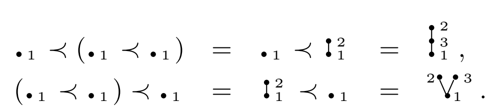
Ainsi, dans tout les cas, $(\mathbf{B})_+$ est stable par l'opération $\prec$.

Remarque. Soient $n \geq 1$ et $T_1, \dots, T_n$ n arbres non vides $\in (\mathbf{B})_+$. Alors
$$T_1 \prec B^-(T_2 \dots T_n) = B^+(T_1 \dots T_n).$$

Le lemme suivant est évident :

Lemme 113 *Étant donné $F, G, H \in (\mathbf{B})_+$,*
$$F \prec (GH) = (F \prec G) \prec H, \tag{4.9}$$
$$F(G \prec H) = (FG) \prec H. \tag{4.10}$$

Remarque. Comme pour $\succ$, $\prec$ n'est pas associative. Par exemple,



Ainsi, grâce au lemme 113, $(\mathbf{B})_+$ muni de la concaténation et de $\prec$ est une algèbre de greffes à droite.

Notons $(\mathbf{B}_r)_+$ l'algèbre de greffes à droite engendrée par l'élément $\bullet_1$. Alors :

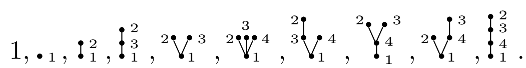
Théorème 114 $(\mathbf{B}_r)_+$ *est libre.*

Preuve. Il faut montrer que si $T_1 \dots T_m$ est une forêt non vide appartenant à $(\mathbf{B}_r)_+$ (avec $m \geq 2$) alors $T_1, \dots, T_m \in (\mathbf{B}_r)_+$. Raisonnons par récurrence sur le degré $n = |T_1|_v + \dots + |T_m|_v$ de la forêt $T_1 \dots T_m$, le cas $n = 1$ étant trivial. Supposons $T_1 \dots T_m \in (\mathbf{B}_r)_+$ de degré $n \geq 2$. Par construction de $\mathbf{B}_r$, il existe $F_1, F_2 \in (\mathbf{B}_r)_+$ tel que $T_1 \dots T_m = F_1 F_2$ ou $T_1 \dots T_m = F_1 \prec F_2$.

1. Si $T_1 \dots T_m = F_1 F_2$, F_1 et F_2 étant non vides, il existe $i \in \{1, \dots, m-1\}$ tel que $F_1 = T_1 \dots T_i$ et $F_2 = T_{i+1} \dots T_m$. Par hypothèse de récurrence, comme $|F_1|_v < n$ et $|F_2|_v < n$, $T_1, \dots, T_i \in (\mathbf{B}_r)_+$ et $T_{i+1}, \dots, T_m \in (\mathbf{B}_r)_+$ ce qui démontre le résultat dans ce cas.
2. Si $T_1 \dots T_m = F_1 \prec F_2$, en notant $G_1 \dots G_k$ la forêt F_1, on a $T_1 \dots T_{m-1} T_m = G_1 \dots G_{k-1}(G_k \prec F_2)$. Nécessairement, $k = m$, $\forall i \in \{1, \dots, m-1\}$, $T_i = G_i$ et $T_m = G_m \prec F_2$. Par hypothèse de récurrence, comme $|F_1|_v < n$, $G_1, \dots, G_m \in (\mathbf{B}_r)_+$. Donc, $\forall i \in \{1, \dots, m-1\}$, $T_i = G_i \in (\mathbf{B}_r)_+$. De plus, $G_m, F_2 \in (\mathbf{B}_r)_+$, donc $T_m = G_m \prec F_2 \in (\mathbf{B}_r)_+$.

Dans tout les cas, $T_1, \dots, T_m \in (\mathbf{B}_r)_+$. On conclut par le principe de récurrence. $\square$

Voici par exemple les arbres appartenant à $\mathbf{B}_r$ de degré ≤ 4 :



Remarquons que $\mathbf{B}_r$ n'est ni une algèbre de Hopf ni une sous-algèbre de $\mathbf{B}^\infty$.

Remarque. Pour toute forêt $F, G, H \in (\mathbf{B})_+$,

$$(F \succ G) \prec H = F \succ (G \prec H). \tag{4.11}$$

On déduit immédiatement de cette remarque et des lemmes 111 et 113 que $(\mathbf{B})_+$ munit de la concaténation, de $\succ$ et de $\prec$ est une algèbre bigreffe. Plus précisément :

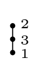
Théorème 115 $(\mathbf{B})_+$ *est engendrée comme algèbre bigreffe par l'élément* $\bullet_1$.

Preuve. Il suffit de montrer que tout les arbres appartenant à $\mathbb{T}$ peuvent être construit à partir de $\bullet_1$ avec les opérations m, $\succ$ et $\prec$. Raisonnons par récurrence sur le degré, le résultat étant évidement vrai au rang 1. Soit $T \in \mathbb{T}$ de degré $n \geq 2$. Il y a deux cas :

1. Si $T = B^-(T_1 \dots T_k)$, avec $T_1, \dots, T_k \in \mathbb{T}$ de degré $< n$ et $k \geq 1$. Par hypothèse de récurrence, $T_1, \dots, T_k$ peuvent être construits à partir de $\bullet_1$ avec les opérations m, $\succ$ et $\prec$. Alors $T = (T_1 \dots T_k) \succ \bullet_1$ peut aussi être construit dans ce cas à partir de $\bullet_1$ avec m, $\succ$ et $\prec$.
2. Si $T = B^+(T_1 \dots T_k)$, avec $T_1, \dots, T_k \in \mathbb{T}$ de degré $< n$ et $k \geq 1$. Toujours par récurrence, $T_1, \dots, T_k$ peuvent être construits à partir de $\bullet_1$ avec m, $\succ$ et $\prec$. Si $k = 1$, $T = B^+(T_1) = T_1 \prec \bullet_1$ et le résultat est démontré. Sinon
$$T = T_1 \prec B^-(T_2 \dots T_k) = T_1 \prec ((T_2 \dots T_k) \succ \bullet_1)$$
et T peut ici encore être construit à partir de $\bullet_1$ avec m, $\succ$ et $\prec$.

Le principe de récurrence permet de conclure. $\square$

Remarque. En comparant la dimension de $\mathbf{B}$ et de l'algèbre bigreffe libre $\mathbf{H}_{BG}$ engendrée par un générateur, on en déduit que $(\mathbf{B})_+$ n'est pas librement engendrée comme algèbre bigreffe par l'élément $\bullet_1$. En effet, on a par exemple,

$$\bullet_1 \prec (\bullet_1 \prec \bullet_1) =$$  $$= \bullet_1 \prec (\bullet_1 \succ \bullet_1).$$

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STRUCTURES HOPF-ALGÉBRIQUES ET OPÉRADIQUES SUR DIFFÉRENTES FAMILLES D'ARBRES.

RÉSUMÉ : Nous introduisons les notions de forêts préordonnées et préordonnées en tas, généralisant les constructions des forêts ordonnées et ordonnées en tas. On démontre que les algèbres des forêts préordonnées et préordonnées en tas sont des algèbres de Hopf pour le coproduit de coupes et on construit un morphisme d'algèbres de Hopf dans l'algèbre des mots tassés. Ensuite, nous définissons un autre coproduit sur les forêts préordonnées donné par la contraction d'arêtes et nous donnons une description combinatoire de morphismes définis sur des algèbres de Hopf de forêts et à valeurs dans les algèbres de Hopf de battages et de battages contractants. Par ailleurs, nous introduisons la notion d'algèbre bigreffe, généralisant les notions d'algèbres de greffes à gauche et à droite. Nous décrivons l'algèbre bigreffe libre engendrée par un générateur et nous munissons cette algèbre d'une structure d'algèbre de Hopf et d'un couplage. Nous étudions ensuite le dual de Koszul de l'operade bigreffe et nous donnons une description combinatoire de l'algèbre bigreffe dual engendrée par un générateur. A l'aide d'une méthode de réécriture, nous prouvons que l'opérade bigreffe est Koszul. Nous définissons la notion de bialgèbre bigreffe infinitésimale et nous prouvons un analogue des théorèmes de Poincaré-Birkhoff-Witt et de Cartier-Milnor-Moore pour les bialgèbres bigreffe infinitésimales connexes. Pour finir, à partir de deux opérateurs de greffes, nous construisons des algèbres de Hopf d'arbres enracinés et ordonnés $\mathbf{B}^i$, $i \in \mathbb{N}^*$, $\mathbf{B}^\infty$ et $\mathbf{B}$ vérifiant les relations d'inclusions $\mathbf{B}^1 \subseteq \ldots \mathbf{B}^i \subseteq \mathbf{B}^{i+1} \subseteq \ldots \subseteq \mathbf{B}^\infty \subseteq \mathbf{B}$. On munit $\mathbf{B}$ d'une structure de bialgèbre dupliciale dendriforme et on en déduit que $\mathbf{B}$ est colibre et auto-duale. Nous démontrons que $\mathbf{B}$ est engendrée comme algèbre bigreffe par un générateur.

HOPF-ALGEBRAICS AND OPERADICS STRUCTURES ON DIFFERENT FAMILIES OF TREES.

ABSTRACT : We introduce the notions of preordered and heap-preordered forests, generalizing the construction of ordered and heap-ordered forests. We prove that the algebras of preordered and heap-preordered forests are Hopf for the cut coproduct, and we construct a Hopf morphism to the Hopf algebra of packed words. In addition, we define another coproduct on the preordered forests given by the contraction of edges, and we give a combinatorial description of morphims defined on Hopf algebras of forests with values in the Hopf algebras of shuffes or quasi-shuffles. Moreover, we introduce the notion of bigraft algebra, generalizing the notions of left and right graft algebras. We describe the free bigraft algebra generated by one generator and we endow this algebra with a Hopf algebra structure, and a pairing. Next, we study the Koszul dual of the bigraft operad and we give a combinatorial description of the free dual bigraft algebra generated by one generator. With the help of a rewriting method, we prove that the bigraft operad is Koszul. We define the notion of infinitesimal bigraft bialgebra and we prove an analogue of Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems for connected infinitesimal bigraft bialgebras. Finally, with two grafting operators, we construct Hopf algebras of rooted and ordered trees $\mathbf{B}^i$, $i \in \mathbb{N}^*$, $\mathbf{B}^\infty$ and $\mathbf{B}$ satisfying the inclusion relations $\mathbf{B}^1 \subseteq \ldots \mathbf{B}^i \subseteq \mathbf{B}^{i+1} \subseteq \ldots \subseteq \mathbf{B}^\infty \subseteq \mathbf{B}$. We endow $\mathbf{B}$ with a structure of duplicial dendriform bialgebra and we deduce that $\mathbf{B}$ is cofree and self-dual. We prove that $\mathbf{B}$ is generated as bigraft algebra by one generator.

DISCIPLINE : Mathématiques.

MOTS-CLÉS : Combinatoires algébriques, Algèbres de Hopf, Arbres, Opérades quadratiques, Dualité de Koszul, Battages et battages contractants.

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